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THE SYMMETRIC INVARIANTS OF THE CENTRALIZERS AND FINITE W-ALGEBRAS

JEAN-YVES CHARBONNEL AND ANNE MOREAU

Abstract. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of rank ℓ over an algebraically closed field $\Bbbk$ of characteristic zero, and let e be a nilpotent element of $\mathfrak{g}$. Denote by $\mathfrak{g}^e$ the centralizer of e in $\mathfrak{g}$ and by $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ the algebra of symmetric invariants of $\mathfrak{g}^e$. We say that e is *good* if the nullvariety of some ℓ homogeneous elements of $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ in $(\mathfrak{g}^e)^*$ has codimension ℓ. If e is good then $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is polynomial. The main result of this paper stipulates that if for some homogeneous generators of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, the initial homogeneous component of their restrictions to $e + \mathfrak{g}^f$ are algebraically independent, with (e, h, f) an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$, then e is good. The proof is strongly based on the theory of finite W-algebras. As applications, we pursue the investigations of [PPY07] and we produce (new) examples of nilpotent elements that verify the above polynomiality condition in simple Lie algebras of both classical and exceptional types. We also give a counter-example in type $\mathbf{D}_7$.

Contents

1. Introduction

1.1. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of rank ℓ over an algebraically closed field $\Bbbk$ of characteristic zero, let $\langle .\,, . \rangle$ be the Killing form of $\mathfrak{g}$ and let G be the adjoint group of $\mathfrak{g}$. If $\mathfrak{a}$ is a subalgebra of $\mathfrak{g}$, we denote by $\mathrm{S}(\mathfrak{a})$ the symmetric algebra of $\mathfrak{a}$. Let $x \in \mathfrak{g}$ and denote by $\mathfrak{g}^x$ and G^x the centralizer of x in $\mathfrak{g}$ and G respectively. Then $\mathrm{Lie}(G^x) = \mathrm{Lie}(G^x_0) = \mathfrak{g}^x$ where G^x_0 denotes the identity component of G^x. Moreover, $\mathrm{S}(\mathfrak{g}^x)$ is a $\mathfrak{g}^x$-module and $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x} = \mathrm{S}(\mathfrak{g}^x)^{G^x_0}$. An interesting question, first raised by A. Premet, is the following:

Question 1. *Is the algebra $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ polynomial algebra in ℓ variables?*

In order to answer this question, thanks to the Jordan decomposition, one can assume that x is nilpotent. Besides, if $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is polynomial for some $x \in \mathfrak{g}$, then it is so for any element in the adjoint orbit $G(x)$ of x. If $x = 0$, it is well-known since Chevalley that $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x} = \mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ is polynomial in ℓ variables. At the

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opposite extreme, if x is a regular nilpotent element of $\mathfrak{g}$, then $\mathfrak{g}^x$ is abelian of dimension ℓ, [DV69], and $S(\mathfrak{g}^x)^{\mathfrak{g}^x} = S(\mathfrak{g}^x)$ is polynomial in ℓ variables too.

For the introduction, let us say most simply that $x \in \mathfrak{g}$ *verifies the polynomiality condition* if $S(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is a polynomial algebra in ℓ variables.

A positive answer to Question 1 was suggested in [PPY07, Conjecture 0.1] for any simple $\mathfrak{g}$ and any $x \in \mathfrak{g}$. O. Yakimova has since discovered a counter-example in type $\mathbf{E}_8$, [Y07], disconfirming the conjecture. More precisely, the elements of the minimal nilpotent orbit in $\mathbf{E}_8$ do not verify the polynomiality condition. The present paper contains another counter-example in type $\mathbf{D}_7$ (cf. Example 7.8). In particular, one cannot expect a positive answer to [PPY07, Conjecture 0.1] for the simple Lie algebras of classical type. Question 1 still remains interesting and is positive for a large number of nilpotent elements $e \in \mathfrak{g}$ as it is explained below.

1.2. We briefly review in this paragraph what has been achieved so far about Question 1. Recall that the *index* of a finite-dimensional Lie algebra $\mathfrak{q}$, denoted by $\operatorname{ind}\mathfrak{q}$, is the minimal dimension of the stabilizers of linear forms on $\mathfrak{q}$ for the coadjoint representation, (cf. [Di74]):

$$\operatorname{ind}\mathfrak{q} := \min\{\dim \mathfrak{q}^{\xi} \ ; \ \xi \in \mathfrak{q}^*\} \ \text{ where } \ \mathfrak{q}^{\xi} := \{x \in \mathfrak{q} \ ; \ \xi([x, \mathfrak{q}]) = 0\}.$$

By [R63], if $\mathfrak{q}$ is algebraic, i.e., $\mathfrak{q}$ is the Lie algebra of some algebraic linear group Q, then the index of $\mathfrak{q}$ is the transcendental degree of the fraction field of Q-invariant rational functions on $\mathfrak{q}^*$. The following result will be important for our purpose.

Theorem 1 ([CM10, Theorem 1.2])**.** *The index of* $\mathfrak{g}^x$ *equals* ℓ *for any* $x \in \mathfrak{g}$.

Theorem 1 was first conjectured by Elashvili in the 90' motivated by a result of Bolsinov, [B91, Theorem 2.1]. It was proven by O. Yakimova when $\mathfrak{g}$ is a simple Lie algebra of classical type, [Y06], and checked by a computer programme by W. de Graaf when $\mathfrak{g}$ is a simple Lie algebra of exceptional type, [DeG08]. Before that, the result was established for some particular classes of nilpotent elements by D.Panyushev, [Pa03].

Theorem 1 is deeply related to Question 1. Indeed, thanks to Theorem 1, [PPY07, Theorem 0.3] applies and by [PPY07, Theorems 4.2 and 4.4], if $\mathfrak{g}$ is simple of type $\mathbf{A}$ or $\mathbf{C}$, then all nilpotent elements of $\mathfrak{g}$ verify the polynomiality condition. The result for the type $\mathbf{A}$ was independently obtained by Brundan and Kleshchev, [BK06]. In [PPY07], the authors also provide some examples of nilpotent elements satisfying the polynomiality condition in the simple Lie algebras of types $\mathbf{B}$ and $\mathbf{D}$, and a few ones in the simple exceptional Lie algebras.

More recently, the analogue question to Question 1 for the positive characteristic was dealt with by L. Topley for the simple Lie algebras of types $\mathbf{A}$ and $\mathbf{C}$, [T12].

1.3. The main goal of this paper is to continue the investigations of [PPY07]. Let us describe the main results. The following definition is central in our work (cf. Definition 3.1):

Definition 1. *An element* $x \in \mathfrak{g}$ *is called a* good element of $\mathfrak{g}$ *if for some homogeneous elements* $p_1, \dots, p_\ell$ *of* $S(\mathfrak{g}^x)^{\mathfrak{g}^x}$*, the nullvariety of* $p_1, \dots, p_\ell$ *in* $(\mathfrak{g}^x)^*$ *has codimension* ℓ *in* $(\mathfrak{g}^x)^*$.

For example, by [PPY07, Theorem 5.4], all nilpotent elements of a simple Lie algebra of type $\mathbf{A}$ are good, and by [Y09, Corollary 8.2], the even nilpotent elements of $\mathfrak{g}$ are good if $\mathfrak{g}$ is of type $\mathbf{B}$ or $\mathbf{C}$ or if $\mathfrak{g}$ is of type $\mathbf{D}$ with odd rank. We rediscover these results in a more general setting (cf. Theorem 5.1 and Corollary 5.8). The good elements verify the polynomiality condition (cf. Proposition 3.2). Moreover, x is good if and only if its nilpotent component in the Jordan decomposition is so (cf. Proposition 3.4).

Let e be a nilpotent element of $\mathfrak{g}$. By the Jacobson-Morosov Theorem, e is embedded into a $\mathfrak{sl}_2$-triple (e, h, f) of $\mathfrak{g}$. Denote by $\mathcal{S}_e := e + \mathfrak{g}^f$ the *Slodowy slice associated with* e. Identify the dual of $\mathfrak{g}$ with $\mathfrak{g}$, and the dual of $\mathfrak{g}^e$ with $\mathfrak{g}^f$, through the Killing form $\langle .\,,.\rangle$. For p in $\mathrm{S}(\mathfrak{g}) \simeq \Bbbk[\mathfrak{g}^*] \simeq \Bbbk[\mathfrak{g}]$, denote by ${}^e p$ the initial homogeneous component of its restriction to $\mathcal{S}_e$. According to [PPY07, Proposition 0.1], if p is in $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, then ${}^e p$ is in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$. The main result of the paper is the following (cf. Theorem 4.1):

Theorem 2. *Suppose that for some homogeneous generators $q_1, \dots, q_\ell$ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, the polynomial functions ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent. Then e is a good element of $\mathfrak{g}$. In particular, $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra and $\mathrm{S}(\mathfrak{g}^e)$ is a free extension of $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$. Moreover, ${}^e q_1, \dots, {}^e q_\ell$ is a regular sequence in $\mathrm{S}(\mathfrak{g}^e)$.*

Theorem 2 applies to a great number of nilpotent orbits in the simple classical Lie algebras (cf. Section 5), and for some nilpotent orbits in the exceptional Lie algebras (cf. Section 6).

To state our results for the simple Lie algebras of types **B** and **D**, let us introduce some more notations. Assume that $\mathfrak{g} = \mathfrak{so}(\mathbb{V}) \subset \mathfrak{gl}(\mathbb{V})$ for some vector space $\mathbb{V}$ of dimension $2\ell + 1$ or 2ℓ. For x an endomorphism of $\mathbb{V}$ and for $i \in \{1, \dots, \dim \mathbb{V}\}$, denote by $Q_i(x)$ the coefficient of degree $\dim \mathbb{V} - i$ of the characteristic polynomial of x. Then for any x in $\mathfrak{g}$, $Q_i(x) = 0$ whenever i is odd. Define a generating family $q_1, \dots, q_\ell$ of the algebra $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ as follows. For $i = 1, \dots, \ell - 1$, set $q_i := Q_{2i}$. If $\dim \mathbb{V} = 2\ell + 1$, set $q_\ell = Q_{2\ell}$ and if $\dim \mathbb{V} = 2\ell$, let q_ℓ be a homogeneous element of degree ℓ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ such that $Q_{2\ell} = q_\ell^2$. Denote by $\delta_1, \dots, \delta_\ell$ the degrees of ${}^e q_1, \dots, {}^e q_\ell$ respectively. By [PPY07, Theorem 2.1], if

$$\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = 0,$$

then the polynomials ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent. In that event, by Theorem 2, e is good and we will say that e is *very good* (cf. Corollary 5.8 and Definition 5.10). The very good nilpotent elements of $\mathfrak{g}$ can be characterized in term of their associated partitions of $\dim \mathbb{V}$ (cf. Lemma 5.11). Theorem 2 also enables to obtain examples of good, but not very good, nilpotent elements of $\mathfrak{g}$; for them, there are a few more work to do (cf. Subsection 5.3).

Thus, we obtain a large number of good nilpotent elements, including all even nilpotent elements in type **B**, or in type **D** with odd rank (cf. Corollary 5.8). For the type **D** with even rank, we obtain the statement for some particular cases (cf. Theorem 5.23).

On the other hand, there are examples of elements that verify the polynomiality condition but that are not good; see Examples 7.5 and 7.6. To deal with them, we use different techniques, more similar to those used in [PPY07]; see Section 7.

As a result of all this, we observe for example that all nilpotent elements of $\mathfrak{so}(\Bbbk^7)$ are good and that all nilpotent elements of $\mathfrak{so}(\Bbbk^n)$, with $n \leqslant 13$, verify the polynomiality condition (cf. Table 5). In particular, by [PPY07, §3.9], this provides examples of good nilpotent elements for which the codimension of $(\mathfrak{g}^e)^*_{\mathrm{sing}}$ in $(\mathfrak{g}^e)^*$ is 1 (cf. Remark 7.7). Here, $(\mathfrak{g}^e)^*_{\mathrm{sing}}$ stands for the set of nonregular linear forms $x \in (\mathfrak{g}^e)^*$, i.e.,

$$(\mathfrak{g}^e)^*_{\mathrm{sing}} := \{x \in (\mathfrak{g}^e)^* \ ; \ \dim (\mathfrak{g}^e)^x > \operatorname{ind} \mathfrak{g}^e = \ell\}.$$

For such nilpotent elements, note that [PPY07, Theorem 0.3] does not apply.

Our results do not cover all nilpotent orbits in type **B** and **D**. As a matter of fact, we obtain a counter-example in type **D** to Premet's conjecture (cf. Example 7.8):

Proposition 1. *The nilpotent elements of $\mathfrak{so}(\Bbbk^{14})$ associated with the partition $(3, 3, 2, 2, 2, 2)$ of 14 do not satisfy the polynomiality condition.*

1.4. The main ingredient to prove Theorem 4.1 is the finite W-algebra associated with the nilpotent orbit $G(e)$ which we emphasize the construction below. Our basic reference for the theory of finite W-algebras is [Pr02]. In the present paper, we refer the reader to Section 4. For i in $\mathbb{Z}$, let $\mathfrak{g}(i)$ be the i-eigenspace of $\operatorname{ad} h$ and set:

$$\mathfrak{p}_+ := \bigoplus_{i\geqslant 0} \mathfrak{g}(i).$$

Then $\mathfrak{p}_+$ is a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{g}^e$. Let $\mathfrak{g}(-1)^0$ be a totally isotropic subspace of $\mathfrak{g}(-1)$ of maximal dimension with respect to the nondegenerate bilinear form

$$\mathfrak{g}(-1) \times \mathfrak{g}(-1) \longrightarrow \Bbbk, \qquad (x, y) \longmapsto \langle e\,, [x, y]\rangle$$

and set:

$$\mathfrak{m} := \mathfrak{g}(-1)^0 \oplus \bigoplus_{i\leqslant -2} \mathfrak{g}(i).$$

Then $\mathfrak{m}$ is a nilpotent subalgebra of $\mathfrak{g}$ with a derived subalgebra orthogonal to e. Denote by $\Bbbk_e$ the one dimensional $\mathrm{U}(\mathfrak{m})$-module defined by the character $x \mapsto \langle e\,, x\rangle$ of $\mathfrak{m}$, denote by $\tilde{Q}_e$ the induced module

$$\tilde{Q}_e := \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{m})} \Bbbk_e$$

and denote by $\tilde{H}_e$ the associative algebra

$$\tilde{H}_e := \mathrm{End}_{\mathfrak{g}}(\tilde{Q}_e)^{\mathrm{op}},$$

known as the *finite W-algebra* associated with e. If $e = 0$, then $\tilde{H}_e$ is isomorphic to the enveloping algebra $\mathrm{U}(\mathfrak{g})$ of $\mathfrak{g}$. If e is a regular nilpotent element, then $\tilde{H}_e$ identifies with the center of $\mathrm{U}(\mathfrak{g})$. More generally, by [Pr02, §6.1], the representation $\mathrm{U}(\mathfrak{g}) \to \mathrm{End}(\tilde{Q}_e)$ is injective on the center $\mathrm{Z}(\mathfrak{g})$ of $\mathrm{U}(\mathfrak{g})$. The algebra $\tilde{H}_e$ is endowed with an increasing filtration, sometimes referred as the *Kazhdan filtration*, and one of the main theorems of [Pr02] states that the corresponding graded algebra is isomorphic to the graded algebra $\mathrm{S}(\mathfrak{g}^e)$. Here, $\mathrm{S}(\mathfrak{g}^e)$ is graded by the *Slodowy grading* (see Subsection 4.1 for more details).

Our idea is to reduce the problem modulo p for a sufficiently big prime integer p, and prove the analogue statement to Theorem 2 in characteristic p. More precisely, we construct in Subsection 4.2 a Lie algebra $\mathfrak{g}_K$ from $\mathfrak{g}$ over an algebraically closed field K of characteristic $p > 0$. The key advantage is essentially that the analogue H_e of the finite W-algebra $\tilde{H}_e$ in this setting is of finite dimension.

1.5. The idea of appealing to the theory of finite W-algebras in this context was initiated in [PPY07, §2]. What is new is to come down to the positive characteristic. More recently, T. Arakawa and A. Premet used affine W-algebras to study an analogue question to Question 1 in the context of jet scheme (private communication). In more detail, assume that $\mathfrak{g}$ is simple of type $\mathbf{A}$ and let e be a nilpotent element of $\mathfrak{g}$. If $\mathfrak{g}_\infty$ denotes the arc space of $\mathfrak{g}$, then Arakawa and Premet show that $\Bbbk[(\mathfrak{g}_e^*)_\infty]^{(\mathfrak{g}_e)_\infty}$ is a polynomial algebra with infinitely many variables. The case where $e = 0$ was already known by Beilinson-Drinfeld, [BD]. Since $\mathfrak{g}$ is of type $\mathbf{A}$, all nilpotent elements of $\mathfrak{g}$ verify the polynomiality condition. Moreover, for any nilpotent element $e \in \mathfrak{g}$, $(\mathfrak{g}^e)^*_{\mathrm{sing}}$ has codimension $\geqslant 3$ in $(\mathfrak{g}^e)^*$ (cf. [Y09, Theorem 5.4]). These two properties are crucial in the proof of Arakawa and Premet.

1.6. The remainder of the paper will be organized as follows.

Section 2 is about general facts on commutative algebra, useful for the Section 3. In Section 3, the notions of good elements and good orbits are introduced, and some properties of good elements are described. Proposition 3.2 asserts that the good elements verify the polynomiality condition. Moreover, Proposition 3.7 gives a sufficient condition for guaranteeing that a given nilpotent element is good. It will be important in Section 4. The main theorem (Theorem 4.1) is stated and proven in Section 4. The proof is based on the theory of finite W-algebras over $\Bbbk$ and over fields of positive characteristic. The section starts with some reminders about this theory following [Pr02]. In Section 5, we give applications of Theorem 4.1 to the simple classical Lie algebras. In Section 6, we give applications to the exceptional Lie algebras of types $\mathbf{E}_6$, $\mathbf{F}_4$ and $\mathbf{G}_2$. This enables us to exhibit a great number of good nilpotent orbits. Other examples, counter-examples, remarks and a conjecture are discussed in Section 7. In this latter section, different techniques are used.

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2. General facts on commutative algebra

We state in this section preliminary results on commutative algebra. Theorem 2.7 will be particularly important in Section 3.

As a rule, for A a homogeneous algebra, A_+ denotes the ideal of A generated by its homogeneous elements of positive degree. Let E be a finite dimensional vector space and let A be a finitely generated homogeneous subalgebra of $\mathrm{S}(E)$. Denote by $\mathcal{N}_0$ the nullvariety of A_+ in E^* and set $N := \dim E - \dim A$.

2.1. Let X be the affine variety $\mathrm{Specm}(A)$ and let π be the morphism from E^* to X whose comorphism is the canonical injection from A into $\mathrm{S}(E)$.

Lemma 2.1. (i) *The irreducible components of the fibers of π have dimension at least N.*

(ii) *If $\mathcal{N}_0$ has dimension N, the fibers of π are equidimensional of dimension N.*

(iii) *If $\mathcal{N}_0$ has dimension N, for some $x_1, \dots, x_N$ in E, the nullvariety of $x_1, \dots, x_N$ in $\mathcal{N}_0$ equals $\{0\}$.*

Proof. (i) Let F be a fiber of π and let U be an open subset of E^* whose intersection with F is not empty and irreducible. The restriction of π to U is a dominant morphism from U to X. So, N is the minimal dimension of the fibers of the restriction of π to U, whence the assertion.

(ii) Denote by x_0 the element A_+ of X. Since A is a homogeneous algebra, there exists a regular action of the one dimensional multiplicative group $\mathrm{G_m}$ on X. Furthermore, for all x in X, x_0 is in the closure of $\mathrm{G_m}.x$. Hence the dimension of the fiber of π at x is at most $\dim \mathcal{N}_0$. As a result, when $\dim \mathcal{N}_0$ is the minimal dimension of the fibers of π, all fiber of π is equidimensional of dimension N by (i).

(iii) For $x = (x_i,\ i \in I)$ a family of elements of E, denote by $A[x]$ the subalgebra of $\mathrm{S}(E)$ generated by A and x, and denote by $\mathcal{N}_0(x)$ its nullvariety in $\mathcal{N}_0$. Since $\mathcal{N}_0$ is a cone, $\mathcal{N}_0(x)$ equals $\{0\}$ if it has dimension 0. So it suffices to find N elements $x_1, \dots, x_N$ of E such that $\mathcal{N}_0(x_1, \dots, x_N)$ has dimension 0. Let us prove by induction on i that for $i = 1, \dots, N$, there exist i elements $x_1, \dots, x_i$ of E such that $\mathcal{N}_0(x_1, \dots, x_i)$ has dimension $N - i$. By induction on i, with $A[x_1, \dots, x_i]$ instead of A, it suffices to find x in E such that $\mathcal{N}_0(x)$ has dimension $N - 1$.

Let $Z_1, \dots, Z_m$ be the irreducible components of $\mathcal{N}_0$ and let I_i be the ideal of definition of Z_i in $\mathrm{S}(E)$. By (i), for $i = 1, \dots, m$, Z_i has dimension N. In particular, I_i does not contain E. So, there exists x in E not in the union of $I_1, \dots, I_m$. Then, for $i = 1, \dots, m$, the nullvariety of x in Z_i is equidimensional of dimension $N-1$. As a result, the nullvariety of the ideal of $\mathrm{S}(E)$ generated by A_+ and x is equidimensional of dimension $N - 1$, whence the assertion. □

For M a graded A-module, set $M_+ := A_+M$.

Lemma 2.2. *Let M be a graded A-module and let V be a homogeneous subspace of M such that $M = V \oplus M_+$. Denote by τ the canonical map $A \otimes_{\Bbbk} V \longrightarrow M$. Then τ is surjective. Moreover, τ is bijective if and only if M is a flat A-module.*

Proof. Let M' be the image of τ. Then by induction on k,

$$M \subset M' + A_+^k M.$$

Since V is homogeneous, M' is homogeneous. So M is contained in M'.

If τ is bijective, then all basis of V is a basis of the A-module M. In particular, it is a flat A-module. Conversely, let us suppose that M is a flat A-module. So, from the exact sequence

$$0 \longrightarrow A_+ \longrightarrow A \longrightarrow \Bbbk \longrightarrow 0$$

one deduces the exact sequence

$$0 \longrightarrow M \otimes_A A_+ \longrightarrow M \longrightarrow \ M \otimes_A k \longrightarrow 0.$$

In particular, the canonical map

$$M \otimes_A A_+ \longrightarrow M$$

is injective. Hence all basis of V is free over A, whence the lemma. □

Proposition 2.3. *Let us consider the following conditions on A:*

1) *A is a polynomial algebra,*
2) *A is a regular algebra,*
3) *A is a polynomial algebra generated by* $\dim A$ *homogeneous elements,*
4) *the A-module* $\mathrm{S}(E)$ *is faithfully flat,*
5) *the A-module* $\mathrm{S}(E)$ *is flat,*
6) *the A-module* $\mathrm{S}(E)$ *is free.*

(i) *The conditions* (1), (2), (3) *are equivalent.*

(ii) *The conditions* (4), (5), (6) *are equivalent. Moreover, Condition* (4) *implies Condition* (2) *and, in that event, $\mathcal{N}_0$ is equidimensional of dimension N.*

(iii) *If $\mathcal{N}_0$ is equidimensional of dimension N, then the conditions* (1), (2), (3), (4), (5), (6) *are all equivalent.*

Proof. Let d be the dimension of A.

(i) The implications (1) ⇒ (2), (3) ⇒ (1) are straigthforward. Let us suppose that A is a regular algebra. Since A is homogeneous and finitely generated, there exists a homogeneous sequence $x_1, \ldots, x_d$ in A_+ representing a basis of A_+/A_+^2. Let A' be the subalgebra of A generated by $x_1, \ldots, x_d$. Then

$$A_+ \subset A' + A_+^2.$$

So by induction on m,

$$A_+ \subset A' + A_+^m$$

for all positive integer m. Since A is homogeneous, $A = A'$ and A is a polynomial algebra generated by d homogeneous elements.

(ii) The implications (4) ⇒ (5), (6) ⇒ (5) are straightforward and (5) ⇒ (6) is a consequence of Lemma 2.2.

(5) ⇒ (4): Recall that $x_0 = A_+$. Let us suppose that $\mathrm{S}(E)$ is a flat A-module. Then π is an open morphism whose image contains x_0. Moreover, $\pi(E^*)$ is stable under the action of $\mathrm{G_m}$. So π is surjective. Hence by [Ma86, Ch. 3, Thm. 7.2], $\mathrm{S}(E)$ is a faithfully flat extension of A.

(4) ⇒ (2): Since $\mathrm{S}(E)$ is regular and since $\mathrm{S}(E)$ is a faithfully flat extension of A, all finitely generated A-module has finite projective dimension. So by [Ma86, Ch. 7,§19,Lemma 2], the global dimension of A is finite. Hence by [Ma86, Ch. 7, Thm. 19.2], A is regular.

If Condition (4) holds, by [Ma86, Ch. 5,Thm. 15.1], the fibers of π are equidimensional of dimension N. So $\mathcal{N}_0$ is equidimensional of dimension N.

(iii) Let us suppose that $\mathcal{N}_0$ is equidimensional of dimension N. By (i) and (ii), it suffices to prove (2) $\Rightarrow$ (5). By Lemma 2.1,(ii) the fibers of π are equidimensional of dimension N. Hence by [Ma86, Ch. 8, Thm. 23.1], $\mathrm{S}(E)$ is a flat extension of A since $\mathrm{S}(E)$ and A are regular. □

2.2. Let $\overline{A}$ be the algebraic closure of A in $\mathrm{S}(E)$.

Lemma 2.4. *Suppose that* $\dim \mathcal{N}_0 = N$. *Then $\mathcal{N}_0$ is the nullvariety of $\overline{A}_+$ in E^*.*

Proof. Let p be a homogeneous element of $\overline{A}$ of positive degree and set $B := A[p]$. Then B is a homogenous algebra having the dimension of A. Denoting by π_B the morphism $E^* \rightarrow \mathrm{Specm}(B)$ whose comorphism is the canonical injection from B into $\mathrm{S}(E)$, the irreducible components of the fibers of π_B have dimension at least N by Lemma 2.1,(i). Since the fiber of π_B at the ideal of augmentation of B is the the nullvariety of p in $\mathcal{N}_0$ and since $\mathcal{N}_0$ has dimension N, $\mathcal{N}_0$ is contained in the nullvariety of p in E^*, whence the lemma. □

Corollary 2.5. *Suppose that* $\dim \mathcal{N}_0 = N$. *Then $\overline{A}$ is the integral closure of A in* $\mathrm{S}(E)$. *In particular, $\overline{A}$ is finitely generated.*

Proof. Since A is a finitely generated homogeneous subalgebra of $\mathrm{S}(E)$, the integral closure of A in $\mathrm{S}(E)$ is so by [Ma86, §33, Lem. 1]. So, one can suppose that A is integrally closed in $\mathrm{S}(E)$. Let p be a homogeneous element of positive degree of $\overline{A}$ and set $B := A[p]$. Denote by π_B and ν the morphisms whose comorphisms are the canonical injections

$$B \longrightarrow \mathrm{S}(E) \text{ and } A \longrightarrow B$$

respectively, whence a commutative diagram

$$\begin{tikzcd} E^* \arrow[rr, "\pi_B"] \arrow[dr, "\pi"'] & & \mathrm{Specm}(B) \arrow[dl, "\nu"] \\ & X & \end{tikzcd} .$$

Since B is a homogeneous subalgebra of $\mathrm{S}(E)$, there exists an action of $\mathrm{G_m}$ on $\mathrm{Specm}(B)$ such that ν is $\mathrm{G_m}$-equivariant. According to Lemma 2.4, the fiber of ν at $x_0 = A_+$ equals B_+. As a result, the fibers of ν are finite. Since B and A have the same fraction field, ν is birational. Hence by Zariski's main theorem [Mu88], ν is an open immersion from $\mathrm{Specm}(B)$ into X. So, ν is surjective since x_0 is in the image of ν and since it is in the closure of all $\mathrm{G_m}$-orbit in X. As a result, ν is an isomorphism and p is in A, whence the corollary since A is homogeneous. □

2.3. Denote by K and $K(E)$ the fraction fields of A and $\mathrm{S}(E)$ respectively.

Lemma 2.6. *Suppose that* $\dim \mathcal{N}_0 = N$ *and suppose that A is a polynomial algebra. Let $v_1, \ldots, v_N$ be a sequence of elements of E such that its nullvariety in $\mathcal{N}_0$ equals* $\{0\}$. *Set $C := \overline{A}[v_1, \ldots, v_N]$.*

(i) *The algebra C is integrally closed and* $\mathrm{S}(E)$ *is the integral closure of C in $K(E)$.*

(ii) *The algebra $\overline{A}$ is Cohen-Macaulay.*

(iii) *The A-module $\overline{A}$ is free and finitely generated.*

Proof. The sequence $v_1, \ldots, v_N$ does exist by Lemma 2.1,(iii).

(i) Since $\overline{A}$ has dimension $\dim E - N$ and since the nullvariety of $v_1, \ldots, v_N$ in $\mathcal{N}_0$ is $\{0\}$, $v_1, \ldots, v_N$ are algebraically independent over A and $\overline{A}$. By Serre's normality criterion [Ma86, Ch. 7, Thm. 19.2], any polynomial algebra over a normal ring is normal. So C is integrally closed since $\overline{A}$ is so by definition.

Moreover, C is a finitely generated homogeneous subalgebra of $\mathrm{S}(E)$ since $\overline{A}$ is too by Corollary 2.5. Since C has dimension $\dim E$, $\mathrm{S}(E)$ is algebraic over C. Then, by Corollary 2.5, $\mathrm{S}(E)$ is the integral closure of C in $K(E)$ since $\mathrm{S}(E)$ is integrally closed as a polynomial algebra and since $\{0\}$ is the nullvariety of C_+ in E^*.

(ii) According to Proposition 2.3, A is generated by homogeneous polynomials $p_1, \ldots, p_d$ with $d := \dim A$. Then $\mathcal{N}_0$ is the nullvariety of $p_1, \ldots, p_d$ in E^* so that $p_1, \ldots, p_d$ is a regular sequence in $\mathrm{S}(E)$ by [Ma86, Ch. 6, Thm. 17.4]. Denoting by K_1 the fraction field of C, the trace map of K over K_1 induces a projection of the C-module $\mathrm{S}(E)$ onto C since $\mathrm{S}(E)$ is the integral closure of C in $K(E)$ by (i). Denote by $a \mapsto a^{\#}$ this projection. For $i = 1, \ldots, d-1$ and for a in $\overline{A}$ such that ap_{i+1} is in the ideal of $\overline{A}$ generated by $p_1, \ldots, p_i$, there exist $b_1, \ldots, b_i$ in $\mathrm{S}(E)$ such that

$$a = b_1 p_1 + \cdots + b_i p_i$$

whence

$$a = b_1^{\#} p_1 + \cdots + b_i^{\#} p_i.$$

Since the nullvariety of $v_1, \ldots, v_N$ in $\mathcal{N}_0$ equals $\{0\}$, $v_1, \ldots, v_N$ are algebraically independent over $\overline{A}$ and $b_1^{\#}, \ldots, b_i^{\#}$ are polynomials in $v_1, \ldots, v_N$ with coefficients in $\overline{A}$. Hence,

$$a = b_1^{\#}(0) p_1 + \cdots + b_i^{\#}(0) p_i$$

since $a, p_1, \ldots, p_i$ are in $\overline{A}$. As a result, $p_1, \ldots, p_d$ is a regular sequence in $\overline{A}$ and $\overline{A}$ is Cohen-Macaulay.

(iii) The algebras A and $\overline{A}$ are graded and $\overline{A}/A_+\overline{A}$ has dimension 0. Moreover, A is regular since it is polynomial. Hence by (ii) and by [Ma86, Ch. 8, Thm. 23.1], $\overline{A}$ is a flat extension of A. So, by Lemma 2.2, $\overline{A}$ is a free extension of A. □

Theorem 2.7. *Suppose that* $\dim \mathcal{N}_0 = N$ *and that* A *is a polynomial algebra. Then* $\overline{A}$ *is a polynomial algebra. Moreover,* $\mathrm{S}(E)$ *is a free extension of* $\overline{A}$*.*

Proof. By Corollary 2.5, $\overline{A}$ is the integral closure of A in $\mathrm{S}(E)$. Let $v_1, \ldots, v_N$, C be as in Lemma 2.6. Let V be a homogeneous complement of $\mathrm{S}(E)C_+$ in $\mathrm{S}(E)$ and let W be a homogeneous complement of $\overline{A}A_+$ in $\overline{A}$. Denote by $\{x_i,\ i \in I\}$ and $\{y_j,\ j \in J\}$ some homogeneous basis of V and W respectively. By Lemma 2.2, V generates the C-module $\mathrm{S}(E)$. Hence there exists a subset L of I such that $\{x_i, i \in L\}$ is a basis of the K_1-space $K(E)$ with K_1 the fraction field of C. By Lemma 2.2 and Lemma 2.6,(iii), $\{y_j, j \in J\}$ is a basis of the free A-module $\overline{A}$. Hence $\{y_j, j \in J\}$ is a basis of the free $A[v_1, \ldots, v_N]$-module C. So $\{x_i y_j,\ (i, j) \in L \times J\}$ is linearly free over $A[v_1, \ldots, v_N]$ since the elements x_i, $i \in L$ are linearly free over C. By Proposition 2.3,(iii), $\mathrm{S}(E)$ is a free extension of $A[v_1, \ldots, v_N]$. So by Lemma 2.2, there exists a homogeneous subspace V' of $\mathrm{S}(E)$ containing $x_i y_j$ for all (i, j) in $L \times J$ such that the canonical map

$$V' \otimes_{\Bbbk} A[v_1, \ldots, v_N] \longrightarrow \mathrm{S}(E)$$

is an isomorphism. Moreover, $\dim V'$ is the degree of the algebraic extension $K(E)$ of $K(v_1, \ldots, v_N)$. The degree of the algebraic extension $K(E)$ of K_1 equals $|L|$ and K_1 is an algebraic extension of $K(v_1, \ldots, v_N)$ whose degree is the degree of the algebraic extension K' of K with K' the fraction field of $\overline{A}$. This degree equals $|J|$ since $\{y_j, j \in J\}$ is a basis of the A-module $\overline{A}$. Hence $\dim V' = |L||I|$. So $\{x_i y_j, (i, j) \in L \times J\}$ is a basis of V'. Hence $\mathrm{S}(E)$ is a free C-module and $\{x_i, i \in L\}$ is a basis. As a result, C is a polynomial algebra by Proposition 2.3 since it is homogeneous. Since C is a faithfully flat extension of $\overline{A}$, $\overline{A}$ is a polynomial algebra by Proposition 2.3 since it is homogeneous. According to Lemma 2.6, $\mathcal{N}_0$ is the nullvariety of $\overline{A}_+$ in E^*. So, by Proposition 2.3,(iii), $\mathrm{S}(E)$ is a free $\overline{A}$-module. □

3. Good elements and good orbits

Recall that $\Bbbk$ is an algebraically closed field of characteristic zero. As in the introduction, $\mathfrak{g}$ is a simple Lie algebra over $\Bbbk$ of rank ℓ, $\langle .\,,.\rangle$ denotes the Killing form of $\mathfrak{g}$, and G its adjoint group.

3.1. The notions of good element and good orbit in $\mathfrak{g}$ are introduced in this paragraph.

Definition 3.1. An element $x \in \mathfrak{g}$ is called a *good element of* $\mathfrak{g}$ if for some homogeneous elements $p_1, \ldots, p_\ell$ of $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$, the nullvariety of $p_1, \ldots, p_\ell$ in $(\mathfrak{g}^x)^*$ has codimension ℓ in $(\mathfrak{g}^x)^*$. A G-orbit in $\mathfrak{g}$ is called *good* if it is the orbit of a good element.

Since the nullvariety of $\mathrm{S}(\mathfrak{g})_+^{\mathfrak{g}}$ in $\mathfrak{g}$ is the nilpotent cone of $\mathfrak{g}$, 0 is a good element of $\mathfrak{g}$. For (g, x) in $G \times \mathfrak{g}$ and for a in $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$, $g(a)$ is in $\mathrm{S}(\mathfrak{g}^{g(x)})^{\mathfrak{g}^{g(x)}}$. So, an orbit is good if and only if all its elements are good.

Denote by K_x the fraction field of $\mathrm{S}(\mathfrak{g}^x)$.

Proposition 3.2. *Let x be a good element of $\mathfrak{g}$. Then $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is a polynomial algebra and $\mathrm{S}(\mathfrak{g}^x)$ is a free $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$-module. Moreover, $K_x^{\mathfrak{g}^x}$ is the fraction field of $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$.*

Proof. Let $p_1, \ldots, p_\ell$ be homogeneous elements of $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ such that the nullvariety of $p_1, \ldots, p_\ell$ in $(\mathfrak{g}^x)^*$ has codimension ℓ. Denote by A the subalgebra of $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ generated by $p_1, \ldots, p_\ell$. Then A is a graded subalgebra of $\mathrm{S}(\mathfrak{g})$ and the nullvariety of A_+ in $(\mathfrak{g}^x)^*$ has codimension ℓ. So, by Lemma 2.1,(ii), A has dimension ℓ. Hence $p_1, \ldots, p_\ell$ are algebraically independent and A is a polynomial algebra. According to [CM10, Thm. 1.2], the index of $\mathfrak{g}^x$ equals ℓ. So, by [R63], the transcendance degree of $K_x^{\mathfrak{g}^x}$ over $\Bbbk$ equals ℓ. Then, since A has dimension ℓ, $K_x^{\mathfrak{g}^x}$ is an algebraic extension of the fraction field of A. As a result, $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is the algebraic closure of A in $\mathrm{S}(\mathfrak{g}^x)$. So, by Theorem 2.7, $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is a polynomial algebra and $\mathrm{S}(\mathfrak{g}^x)$ is a free $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$-module. Since $K_x^{\mathfrak{g}^x}$ is an algebraic extension of the fraction field of A, for p in $K_x^{\mathfrak{g}^x}$, ap verifies an integral dependence equation over $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ for some a in $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$. Then, since $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is integrally closed in K_x, $K_x^{\mathfrak{g}^x}$ is the fraction field of $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$. □

Remark 3.3. The algebra $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ may be polynomial with x not good. Indeed, let us consider a nilpotent element e of $\mathfrak{g} = \mathfrak{so}(\Bbbk^{10})$ associated with the partition $(3, 3, 2, 2)$. The algebra $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is polynomial, generated by elements of degrees $1, 1, 2, 2, 5$. But the nullcone has an irreducible component of codimension at most 4. So, e is not good; see Example 7.5 in Section 7 for more details.

For $x \in \mathfrak{g}$, denote by x_{s} and x_{n} the semisimple and the nilpotent components of x respectively.

Proposition 3.4. *Let x be in $\mathfrak{g}$. Then x is good if and only if x_{n} is a good element of the derived algebra of $\mathfrak{g}^{x_{\mathrm{s}}}$.*

Proof. Let $\mathfrak{z}$ be the center of $\mathfrak{g}^{x_{\mathrm{s}}}$ and let $\mathfrak{a}$ be the derived algebra of $\mathfrak{g}^{x_{\mathrm{s}}}$. Then

$$\mathfrak{g}^x = \mathfrak{z} \oplus \mathfrak{a}^{x_{\mathrm{n}}}, \qquad \mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x} = \mathrm{S}(\mathfrak{z}) \otimes_\Bbbk \mathrm{S}(\mathfrak{a}^{x_{\mathrm{n}}})^{\mathfrak{a}^{x_{\mathrm{n}}}}.$$

By the first equality, $(\mathfrak{a}^{x_{\mathrm{n}}})^*$ identifies with the orthogonal complement of $\mathfrak{z}$ in $(\mathfrak{g}^x)^*$. Set $d := \dim \mathfrak{z}$. Suppose that x_{n} is a good element of $\mathfrak{a}$. Let $p_1, \ldots, p_{\ell-d}$ be homogeneous elements of $\mathrm{S}(\mathfrak{a}^{x_{\mathrm{n}}})^{\mathfrak{a}^{x_{\mathrm{n}}}}$ whose nullvariety in $(\mathfrak{a}^{x_{\mathrm{n}}})^*$ has codimension $\ell - d$. Denoting by $v_1, \ldots, v_d$ a basis of $\mathfrak{z}$, the nullvariety of $v_1, \ldots, v_d, p_1, \ldots, p_{\ell-d}$ in $(\mathfrak{g}^x)^*$ is the nullvariety of $p_1, \ldots, p_{\ell-d}$ in $(\mathfrak{a}^{x_{\mathrm{n}}})^*$. Hence, x is a good element of $\mathfrak{g}$.

Conversely, let us suppose that x is a good element of $\mathfrak{g}$. By Proposition 3.2, $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is a polynomial algebra generated by homogeneous polynomials $p_1, \ldots, p_\ell$. Since $\mathfrak{z}$ is contained in $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$, $p_1, \ldots, p_\ell$ can be chosen so that $p_1, \ldots, p_d$ are in $\mathfrak{z}$ and $p_{d+1}, \ldots, p_d$ are in $\mathrm{S}(\mathfrak{a}^{x_{\mathrm{n}}})^{\mathfrak{a}^{x_{\mathrm{n}}}}$. Then the nullvariety of $p_{d+1}, \ldots, p_d$ in $(\mathfrak{a}^{x_{\mathrm{n}}})^*$ has codimension $\ell - d$. Hence, x_{n} is a good element of $\mathfrak{a}$. □

3.2. Let e be a nilpotent element of $\mathfrak{g}$, embedded into an $\mathfrak{sl}_2$-triple (e,h,f) of $\mathfrak{g}$. Identify the dual of $\mathfrak{g}$ with $\mathfrak{g}$, and the dual of $\mathfrak{g}^e$ with $\mathfrak{g}^f$ through the Killing form $\langle .\,,.\rangle$. For p in $\mathrm{S}(\mathfrak{g}) \simeq \Bbbk[\mathfrak{g}]$, denote by $\kappa(p)$ the restriction to $\mathfrak{g}^f$ of the polynomial function $x \mapsto p(e+x)$ and denote by ${}^e p$ its initial homogeneous component. According to [PPY07, Prop. 0.1], for p in $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, ${}^e p$ is in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$.

Let I be the ideal of $\mathrm{S}(\mathfrak{g}^e)$ generated by the elements $\kappa(p)$, for p running through $\mathrm{S}_+(\mathfrak{g})^{\mathfrak{g}}$, and set $N := \dim \mathfrak{g}^e - \ell$.

Lemma 3.5. *The nullvariety of I in $\mathfrak{g}^f$ is equidimensional of dimension N.*

Proof. Let $\mathcal{S}_e$ be the Slodowy slice $e + \mathfrak{g}^f$ associated with e, and let θ_e be the map

$$G \times \mathcal{S}_e \longrightarrow \mathfrak{g}, \qquad (g,x) \mapsto g(x).$$

Then θ_e is a smooth G-equivariant morphism onto a G-invariant open subset containing $G(e)$. In particular, it is equidimensional of dimension $\dim \mathcal{S}_e$. Denoting by X the nullvariety of I in $\mathfrak{g}^f$, $G \times (e+X)$ is the inverse image by θ_e of the nipotent cone of $\mathfrak{g}$. Hence, $G \times (e+X)$ is equidimensional of dimension

$$\dim \mathfrak{g} - \ell + \dim \mathcal{S}_e = N + \dim \mathfrak{g}$$

since the nilpotent cone is irreducible of codimension ℓ and contains $G(e)$. The lemma follows. $\square$

The symmetric algebra $\mathrm{S}(\mathfrak{g}^e)$ is naturally graded by the degree of elements. For m a nonnegative integer, denote by $\mathrm{S}^m(\mathfrak{g}^e)$ the homogeneous component of degree m and set:

$$\mathrm{S}_m(\mathfrak{g}^e) := \bigoplus_{i \geqslant m} \mathrm{S}^m(\mathfrak{g}^e).$$

Then $\mathrm{S}_m(\mathfrak{g}^e), m = 0,1,\ldots$ is a decreasing filtration of $\mathrm{S}(\mathfrak{g}^e)$ and its associated graded algebra is the usual graded algebra $\mathrm{S}(\mathfrak{g}^e)$. For J a subquotient of $\mathrm{S}(\mathfrak{g}^e)$, the filtration of $\mathrm{S}(\mathfrak{g}^e)$ induces a filtration of J and its associated graded space is denoted by $\mathrm{gr}(J)$.

Lemma 3.6. *The nullvariety of* $\mathrm{gr}(I)$ *in $\mathfrak{g}^f$ has dimension N.*

Proof. By definition,

$$\mathrm{gr}(\mathrm{S}(\mathfrak{g}^e)/I) = \bigoplus_{l \in \mathbb{N}} \mathrm{S}_l(\mathfrak{g}^e)/(\mathrm{S}_{l+1}(\mathfrak{g}^e) + I \cap \mathrm{S}_l(\mathfrak{g}^e))$$

so that $\mathrm{gr}(\mathrm{S}(\mathfrak{g}^e)/I)$ is the quotient of $\mathrm{S}(\mathfrak{g}^e)$ by $\mathrm{gr}(I)$. According to [Ma86, Thm. 13.4], $\mathrm{gr}(\mathrm{S}(\mathfrak{g}^e)/I)$ and $\mathrm{S}(\mathfrak{g}^e)/I$ have the same dimension, whence the corollary by Lemma 3.5. $\square$

The following proposition will be useful to prove Theorem 4.1 in the next Section. It gives a sufficient condition for guaranteeing that a given nilpotent element is good.

Proposition 3.7. *Let $q_1,\ldots,q_\ell$ be homogeneous generators of* $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ *and let J be the ideal of* $\mathrm{S}(\mathfrak{g}^e)$ *generated by ${}^e q_1 \ldots {}^e q_\ell$. Suppose that for $a_1,\ldots,a_\ell$ in* $\mathrm{S}(\mathfrak{g}^e)$*, the following implication holds:*

$$(\,a_1({}^e q_1) + \cdots + a_\ell({}^e q_\ell) = 0 \quad \Longrightarrow \quad \forall i \in \{1,\ldots,\ell\},\ a_i \in J\,).$$

Then $\mathrm{gr}(I) = J$. *In particular, e is a good element of* $\mathfrak{g}$.

Proof. By definition, J is contained in $\mathrm{gr}(I)$. Let us suppose that J is strictly contained in $\mathrm{gr}(I)$. A contradiction is expected. For a in $\mathrm{S}(\mathfrak{g}^e)$, let $\nu(a)$ be the biggest integer such that a is in $\mathrm{S}_{\nu(a)}(\mathfrak{g}^e)$ and let $\overline{a}$ be the image of a in $\mathrm{gr}(\mathrm{S}(\mathfrak{g}^e))$. For $i = 1,\ldots,\ell$, let d_i be the degree of ${}^e q_i$. For $\mathbf{a} := (a_1,\ldots,a_\ell)$ in $\mathrm{S}(\mathfrak{g}^e)^\ell$, set:

$$\nu(\mathbf{a}) := \inf\{\nu(a_1)+d_1,\ldots,\nu(a_\ell)+d_\ell\}, \qquad \sigma(\mathbf{a}) := a_1\kappa(q_1) + \cdots + a_\ell\kappa(q_\ell).$$

Since J is strictly contained in $\mathrm{gr}(I)$, there is $\mathbf{a} = (a_1, \dots, a_\ell)$ in $\mathrm{S}(\mathfrak{g}^e)^\ell$ such that $\overline{\sigma(\mathbf{a})}$ is not in J. Let d be the degree of $\overline{\sigma(\mathbf{a})}$. Choose such $\mathbf{a}$ in $\mathrm{S}(\mathfrak{g}^e)^\ell$ such that $\nu(\mathbf{a})$ is maximal.

For $i = 1, \dots, \ell$, write

$$a_i = a_{i,0} + a_{i,+}$$

with $a_{i,0}$ homogeneous of degree $\nu(a_i)$ and $\nu(a_{i,+}) > \nu(a_i)$. Let L be the set of indices i such that $\nu(\mathbf{a}) = \nu(a_i) + d_i$. Since $\overline{\sigma(\mathbf{a})}$ is not in J,

$$\sum_{i \in L} a_{i,0}({}^e q_i) = 0.$$

So, by hypothesis, $a_{1,0}, \dots, a_{\ell,0}$ are in J so that

$$\overline{\sum_{i \in L} a_{i,0} \kappa(q_i)} \in J.$$

Moreover,

$$\sigma(\mathbf{a}) = \sum_{i \in L} a_{i,0}\kappa(q_i) + \sigma(\mathbf{b}) \quad \text{with} \quad b_i := \begin{cases} a_{i,+} & \text{if} \quad i \in L \\ a_i & \text{if} \quad i \notin L \end{cases} \quad \text{and} \quad \mathbf{b} = (b_1, \dots, b_\ell).$$

Since $\overline{\sigma(\mathbf{a})}$ has degree d and is not in J, $\overline{\sigma(\mathbf{b})}$ is an element of degree d which is not in J. We have obtained the expected contradiction since $\nu(\mathbf{b}) > \nu(\mathbf{a})$.

As a consequence, $\mathrm{gr}(I) = J$ and the last assertion of the proposition is a straightforward consequence of Lemma 3.6. □

4. Proof of Theorem 2 and finite W-algebras

As in the previous section, $\mathfrak{g}$ is a simple Lie algebra over $\Bbbk$ and (e, h, f) is an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$. The goal of this section is to prove the following theorem (see also Theorem 2).

Theorem 4.1. *Suppose that for some homogeneous generators $q_1, \dots, q_\ell$ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, the polynomial functions ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent. Then e is a good element of $\mathfrak{g}$. In particular, $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra and $\mathrm{S}(\mathfrak{g}^e)$ is a free extension of $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$. Moreover, ${}^e q_1, \dots, {}^e q_\ell$ is a regular sequence in $\mathrm{S}(\mathfrak{g}^e)$.*

To that end, the theory of finite W-algebras will be strongly used. Our main reference for this topic is [Pr02] and the section starts with some notations and results of [Pr02]. The heart of the proof of Theorem 4.1 is presented in Subsection 4.6.

4.1. For i in $\mathbb{Z}$, let $\mathfrak{g}(i)$ be the eigenspace of eigenvalue i of $\mathrm{ad}\, h$ and set:

$$\mathfrak{p}_+ := \bigoplus_{i \geqslant 0} \mathfrak{g}(i).$$

Then $\mathfrak{p}_+$ is a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{g}^e$. So, the bilinear form

$$\mathfrak{g}(-1) \times \mathfrak{g}(-1) \longrightarrow \Bbbk, \qquad (x, y) \longmapsto \langle e\,, [x, y] \rangle$$

is nondegenerate. Let $\mathfrak{g}(-1)^0$ be a totally isotropic subspace of $\mathfrak{g}(-1)$ of maximal dimension and set:

$$\mathfrak{m} := \mathfrak{g}(-1)^0 \oplus \bigoplus_{i \leqslant -2} \mathfrak{g}(i)$$

so that $\mathfrak{m}$ is an ad-nilpotent subalgebra of $\mathfrak{g}$ with the derived subalgebra orthogonal to e. Denote by $\Bbbk_e$ the one dimensional $\mathrm{U}(\mathfrak{m})$-module defined by the character $x \mapsto \langle e\,, x \rangle$ of $\mathfrak{m}$, denote by $\tilde{Q}_e$ the induced module

$$\tilde{Q}_e := \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{m})} \Bbbk_e$$

and denote by $\tilde{H}_e$ the associative algebra

$$\tilde{H}_e := \mathrm{End}_{\mathfrak{g}}(\tilde{Q}_e)^{\mathrm{op}}.$$

By [Pr02, §6.1], the representation $\tilde{\rho}_e : \mathrm{U}(\mathfrak{g}) \to \mathrm{End}(\tilde{Q}_e)$ is injective on the center $\mathrm{Z}(\mathfrak{g})$ of $\mathrm{U}(\mathfrak{g})$.

Let $\{x_1, \ldots, x_m\}$ be a basis of $\mathfrak{p}_+$ such that x_i is an eigenvector of eigenvalue n_i of $\mathrm{ad}\, h$, and let $z_1, \ldots, z_s$ be a basis of a totally isotropic complement to $\mathfrak{g}(-1)^0$ in $\mathfrak{g}(-1)$. For $(\mathbf{i}, \mathbf{j}) = (i_1, \ldots, i_m, j_1, \ldots, j_s)$ in $\mathbb{N}^m \times \mathbb{N}^s$, set:

$$x^{\mathbf{i}} z^{\mathbf{j}} := x_1^{i_1} \cdots x_m^{i_m} z_1^{j_1} \cdots z_s^{j_s} \qquad |(\mathbf{i}, \mathbf{j})|_e := \textstyle\sum_{k=1}^m i_k(n_k + 2) + \sum_{k=1}^s j_k$$

By the PBW theorem, $\{x^{\mathbf{i}} z^{\mathbf{j}} \otimes 1,\ (\mathbf{i}\,\mathbf{j}) \in \mathbb{N}^m \times \mathbb{N}^s\}$ is a basis of $\tilde{Q}_e$. For k in $\mathbb{N}$, let $\tilde{H}_e^k$ be the subspace of elements h of $\tilde{H}_e$ such that $\tilde{\rho}_e(h)(1 \otimes 1)$ is a linear combination of the $x^{\mathbf{i}} z^{\mathbf{j}} \otimes 1$, $|(\mathbf{i}, \mathbf{j})|_e \leqslant k$. Then the sequence $\tilde{H}_e^k, k = 0, 1, \ldots$ is an increasing filtration of the algebra $\tilde{H}_e$.

Recall that $\mathcal{S}_e$ is the Slodowy slice $e + \mathfrak{g}^f$ associated with e. Since $\mathfrak{g}^f$ identifies with the dual of $\mathfrak{g}^e$, the algebra $\Bbbk[\mathcal{S}_e]$ identifies with $\mathrm{S}(\mathfrak{g}^e)$. Denoting by $t \mapsto h(t)$ the one parameter subgroup of G generated by $\mathrm{ad}\, h$, $\mathcal{S}_e$ is invariant by the one parameter subgroup $t \mapsto t^{-2} h(t)$. Hence, this group induces a gradation on the algebra $\mathrm{S}(\mathfrak{g}^e)$. One of the main theorems of [Pr02] says that the graded algebra associated with the filtration of $\tilde{H}_e$ is isomorphic to the so defined graded algebra $\mathrm{S}(\mathfrak{g}^e)$ (see also [GG02] for the case where $\Bbbk = \mathbb{C}$).

4.2. Let h be the Coxeter number of the root system of $\mathfrak{g}$. According to the Bala-Carter theory [C85, Ch. 5], there exists a $\mathbb{Z}$-form $\mathfrak{g}_{\mathbb{Z}}$ of $\mathfrak{g}$ such that (e, h, f) is an $\mathfrak{sl}_2$-triple of the $\mathbb{Q}$-form $\mathfrak{g}_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} \mathfrak{g}_{\mathbb{Z}}$ of $\mathfrak{g}$. Let $d_{\mathbb{Z}}$ be the determinant of the Killing form of $\mathfrak{g}_{\mathbb{Z}}$ in a basis of $\mathfrak{g}_{\mathbb{Z}}$ and let N be a sufficiently big integer such that e, h, f are in $\mathfrak{g}_N := \mathbb{Z}[1/N!] \otimes_{\mathbb{Z}} \mathfrak{g}_{\mathbb{Z}}$, and such that

$$\mathfrak{g}(i) = \Bbbk \otimes_{\mathbb{Z}[1/N!]} (\mathfrak{g}(i) \cap \mathfrak{g}_N), \qquad \mathfrak{g}(-1)^0 = \Bbbk \otimes_{\mathbb{Z}[1/N!]} (\mathfrak{g}(-1)^0 \cap \mathfrak{g}_N),$$

$$N > d_{\mathbb{Z}} \qquad N > h, \qquad N > \langle e\,, f \rangle, \qquad N > \max\{i + 2 \ ; \ \mathfrak{g}(i) \neq \{0\}\},$$

Then, one can choose the elements $x_1, \ldots, x_m, z_1, \ldots, z_s$ of $\mathfrak{g}$ in $\mathfrak{g}_N$. Let p be a prime number bigger than N. Since p is not invertible in $\mathbb{Z}[1/N!]$, p is contained in a maximal ideal $\mathfrak{M}_p$ of $\mathbb{Z}[1/N!]$. Then $\mathbb{Z}[1/N!]/\mathfrak{M}_p$ is an algebraic extension of $\mathbb{F}_p$. Let K be an algebraic closure of $\mathbb{Z}[1/N!]/\mathfrak{M}_p$ and set:

$$\mathfrak{g}_K := K \otimes_{\mathbb{Z}[1/N!]} \mathfrak{g}_N$$

Denote by G_K a simple, simply connected, algebraic K-group such that $\mathfrak{g}_K = \mathrm{Lie}(G_K)$. Since $N > d_{\mathbb{Z}}$, the Killing form of $\mathfrak{g}_N$ induces a nondegenerate bilinear form on $\mathfrak{g}_K$, that we will also denote by $\langle .\,, . \rangle$.

As a Lie algebra of an algebraic group over a field of positive characteristic, $\mathfrak{g}_K$ is a restricted Lie algebra whose p-operation is denoted by $x \mapsto x^{[p]}$. For x semi-simple, $x^{[p]} = x$ and for x nilpotent, $x^{[p]} = 0$ since $p > h$; see for instance [V72, §1]. For χ in $\mathfrak{g}_K^*$, denote by $\mathrm{U}_\chi(\mathfrak{g}_K)$ the quotient of $\mathrm{U}(\mathfrak{g}_K)$ by the ideal generated by the elements $x^p - x^{[p]} - \chi(x)^p$, with $x \in \mathfrak{g}_K$. More generally, if $\mathfrak{a}$ is a restricted subalgebra of $\mathfrak{g}_K$, we denote by $\mathrm{U}_\chi(\mathfrak{a})$ the quotient of $\mathrm{U}(\mathfrak{a})$ by the ideal generated by the elements $x^p - x^{[p]} - \chi(x)^p$, with $x \in \mathfrak{a}$. Then set

$$\mathrm{U}_e(\mathfrak{g}_K) := \mathrm{U}_{\chi_e}(\mathfrak{g}_K) \qquad \text{and} \qquad \mathrm{U}_e(\mathfrak{a}) := \mathrm{U}_{\chi_e}(\mathfrak{a}),$$

where χ_e is the linear form

$$\chi_e \ : \ \mathfrak{g}_K \to K, \qquad x \mapsto \langle x\,, e \rangle.$$

For all $\chi \in \mathfrak{g}_K^*$, the restriction to $\mathfrak{g}_K$ of the quotient map $\mathrm{U}(\mathfrak{g}_K) \to \mathrm{U}_\chi(\mathfrak{g}_K)$ is an embedding and $\mathrm{U}_\chi(\mathfrak{g}_K)$ is a finite dimensional algebra of dimension $p^{\dim \mathfrak{g}}$ by the PBW Theorem. Moreover, for any restricted subalgebra $\mathfrak{a}$ of $\mathfrak{g}_K$, the canonical map $\mathrm{U}(\mathfrak{a}) \to \mathrm{U}_\chi(\mathfrak{g}_K)$ defines through the quotient map an embedding from $\mathrm{U}_\chi(\mathfrak{a})$ into $\mathrm{U}_\chi(\mathfrak{g}_K)$.

Denote by $e, h, f, x_1, \dots, x_m, z_1, \dots, z_s$ the elements $1\otimes e$, $1\otimes h$, $1\otimes f$, $1\otimes x_1, \dots, 1\otimes x_m$, $1\otimes z_1, \dots, 1\otimes z_s$ of $\mathfrak{g}_K$ respectively. Because of the choice of N, for i in $\mathbb{Z}$, the i-eigenspace $\mathfrak{g}_K(i)$ of $\operatorname{ad} h$ in $\mathfrak{g}_K$ verifies

$$\mathfrak{g}_K(i) = K \otimes_{\mathbb{Z}[1/N!]} (\mathfrak{g}(i) \cap \mathfrak{g}_N)$$

Set:

$$\mathfrak{g}_K(-1)^0 := K \otimes_{\mathbb{Z}[1/N!]} (\mathfrak{g}(-1)^0 \cap \mathfrak{g}_N), \qquad \mathfrak{m}_K := \mathfrak{g}_K(-1)^0 \oplus \bigoplus_{i\leqslant -2} \mathfrak{g}_K(i),$$

$$\mathfrak{p}_{+,K} := \bigoplus_{i\geqslant 0} \mathfrak{g}_K(i), \qquad \mathfrak{g}_K(-1)^1 := \operatorname{span}(\{z_1, \dots, z_s\}).$$

Then $\mathfrak{m}_K$ is an ad-nilpotent Lie algebra with a derived algebra orthogonal to e. Moreover, it is a restricted subalgebra of $\mathfrak{g}_K$ whose p-operation equals 0 since $\mathfrak{m}_K$ is ad-nilpotent. Let K_e be the one dimensional $\mathfrak{m}_K$-module defined by the character χ_e of $\mathfrak{m}_K$. Then K_e is a $\mathrm{U}_e(\mathfrak{m}_K)$-module. Denote by Q the induced module

$$Q := \mathrm{U}_e(\mathfrak{g}_K) \otimes_{\mathrm{U}_e(\mathfrak{m}_K)} K_e,$$

and set

$$H := \operatorname{End}_{\mathfrak{g}_K}(Q)^{\mathrm{op}}.$$

Then Q and H are finite dimensional. For k in $\mathbb{N}$, set

$$\Lambda_k := \{(l_1, \dots, l_k),\ l_i \in \mathbb{N},\ 0 \leqslant l_i \leqslant p-1\}.$$

By the PBW Theorem, $\{x^{\mathbf{i}} z^{\mathbf{j}}\otimes 1,\ (\mathbf{i}, \mathbf{j}) \in \Lambda_m \times \Lambda_s\}$ is a basis of Q. For h in H, h is determined by its value at $1\otimes 1$,

$$h(1\otimes 1) = \sum_{(\mathbf{i},\mathbf{j})\in\Lambda_m\times\Lambda_s} a_{\mathbf{i},\mathbf{j}}\, x^{\mathbf{i}} z^{\mathbf{j}}\otimes 1,$$

with the $a_{\mathbf{i},\mathbf{j}}$'s in K. Denote by $n(h)$ the biggest integers $|(\mathbf{i}, \mathbf{j})|_e$ with $(\mathbf{i}, \mathbf{j}) \in \Lambda_m \times \Lambda_s$ such that $a_{\mathbf{i},\mathbf{j}} \neq 0$. For k in $\mathbb{N}$, denote by H^k the linear vector space spanned by the elements h of H such that $n(h) \leqslant k$. By [Pr02, 3.3], the sequence $H^0, H^1, \dots$ is an increasing filtration of the algebra H.

4.3. According to [V72, Prop. 2.1], the algebra $\mathrm{U}(\mathfrak{g}_K)^{G_K}$ of the invariant elements of the adjoint action of G_K in $\mathrm{U}(\mathfrak{g}_K)$ is a polynomial algebra generated by some elements $T_1, \dots, T_\ell$ of the augmentation ideal of $\mathrm{U}(\mathfrak{g}_K)$.

Let Z_K be the center of $\mathrm{U}(\mathfrak{g}_K)$ and let Z_0 be the subalgebra of $\mathrm{U}(\mathfrak{g}_K)$ generated by the elements $x^p - x^{[p]}$, with x in $\mathfrak{g}_K$. Then Z_0 is a polynomial algebra contained in Z_K and, by [V72, Thm. 3.1],

$$Z_K = Z_0[T_1, \dots, T_\ell]. \tag{1}$$

For $\mathbf{i} = (i_1, \dots, i_\ell)$ in $\mathbb{N}^\ell$, set

$$|\mathbf{i}| := i_1 + \cdots + i_\ell, \qquad T^{\mathbf{i}} := T_1^{i_1} \cdots T_\ell^{i_\ell}.$$

By [V72, Thm. 3.1], Z_K is a free Z_0-module with basis $\{T^{\mathbf{i}},\ \mathbf{i} \in \Lambda_\ell\}$.

Let χ be in $\mathfrak{g}_K^*$. Denote by $Z_{K,\chi}$ the image of Z_K by the quotient morphism $\mathrm{U}(\mathfrak{g}_K) \to \mathrm{U}_\chi(\mathfrak{g}_K)$, and by I_χ the ideal of $Z_{K,\chi}$ generated by the images of $T_1, \dots, T_\ell$ in $Z_{K,\chi}$.

Lemma 4.2. *Let χ be in $\mathfrak{g}_K^*$.*

(i) *The ideal I_χ of $Z_{K,\chi}$ is strictly contained in $Z_{K,\chi}$. Moreover, $\{T^{\mathbf{i}}\,;\ \mathbf{i} \in \Lambda_\ell,\ |\mathbf{i}| \geqslant m\}$ is a basis of I_χ^m.*

(ii) *For m nonnegative integer, the dimensions of the K-spaces I_χ^m and $\mathrm{U}_\chi(\mathfrak{g}_K) I_\chi^m$ do not depend on χ.*

Proof. (i) Let E be the K-subspace of Z_K generated by the elements $T^{\mathbf{i}}$, $\mathbf{i} \in \Lambda_\ell$. Since $p > h$, the restriction to E of the quotient map $\mathrm{U}(\mathfrak{g}_K) \to \mathrm{U}_\chi(\mathfrak{g}_K)$ is an embedding and its image is $Z_{K,\chi}$. Identifying E with $Z_{K,\chi}$, I_χ is the subspace of $Z_{K,\chi}$ generated by the elements $T^{\mathbf{i}}$, $\mathbf{i} \in \Lambda_\ell \smallsetminus \{0\}$. So, it is strictly contained in $Z_{K,\chi}$. Moreover, $\{T^{\mathbf{i}} \,;\, \mathbf{i} \in \Lambda_\ell,\ |\mathbf{i}| \geqslant m\}$ is a basis of I_χ^m.

(ii) Let $\{y_1, \dots, y_n\}$ be a basis of $\mathfrak{g}_K$. For $\mathbf{i} = (i_1, \dots, i_n)$ in $\mathbb{N}^n$, set:

$$y^{\mathbf{i}} := y_1^{i_1} \cdots y_n^{i_n}.$$

The Z_0-module $\mathrm{U}(\mathfrak{g}_K)$ is free with basis $\{y^{\mathbf{i}},\ \mathbf{i} \in \Lambda_n\}$. Let F be the subspace of $\mathrm{U}(\mathfrak{g}_K)$ generated by the elements $y^{\mathbf{i}} T^{\mathbf{j}}$ with $(\mathbf{i}, \mathbf{j}) \in \Lambda_n \times \Lambda_\ell$ and $|\mathbf{j}| \geqslant m$. Then the restriction to F of the quotient map $\mathrm{U}(\mathfrak{g}_K) \to \mathrm{U}_\chi(\mathfrak{g}_K)$ is a surjective morphism onto $\mathrm{U}_\chi(\mathfrak{g}_K) I_\chi^m$.

Let d be the dimension of $\mathrm{U}_0(\mathfrak{g}_K) I_0^m$. Choose $(\mathbf{i}_1, \mathbf{j}_1), \dots, (\mathbf{i}_d, \mathbf{j}_d)$ such that the elements

$$y^{\mathbf{i}_1} T^{\mathbf{j}_1}, \dots, y^{\mathbf{i}_d} T^{\mathbf{j}_d}$$

of F induce a basis of $\mathrm{U}_0(\mathfrak{g}_K) I_0^m$. The usual filtration on $\mathrm{U}(\mathfrak{g}_K)$ induces filtrations on $\mathrm{U}_0(\mathfrak{g}_K)$ and $\mathrm{U}_\chi(\mathfrak{g}_K)$ having the same associated graded spaces. Indeed, for $x \in \mathfrak{g}_K$, the images of the elements $x^p - x^{[p]}$ and $x^p - x^{[p]} - \chi(x)^p$ are the same in the associated graded spaces $\mathrm{gr}(\mathrm{U}_0(\mathfrak{g}_K))$ and $\mathrm{gr}(\mathrm{U}_\chi(\mathfrak{g}_K))$. The images of $y^{\mathbf{i}_1} T^{\mathbf{j}_1}, \dots, y^{\mathbf{i}_d} T^{\mathbf{j}_d}$ in the graded space $\mathrm{gr}(\mathrm{U}_0(\mathfrak{g}_K))$ are linearly free. Hence, the images of $y^{\mathbf{i}_1} T^{\mathbf{j}_1}, \dots, y^{\mathbf{i}_d} T^{\mathbf{j}_d}$ in $\mathrm{gr}(\mathrm{U}_\chi(\mathfrak{g}_K))$ are linearly free too. As a result, $\mathrm{U}_\chi(\mathfrak{g}_K) I_\chi^m$ has dimension at least d.

Exchanging the roles of $\mathrm{U}_\chi(\mathfrak{g}_K) I_\chi^m$ and $\mathrm{U}_0(\mathfrak{g}_K) I_0^m$ in the above lines of arguments, we obtain that $\mathrm{U}_\chi(\mathfrak{g}_K) I_\chi^m$ has dimension at most d, whence the assertion. □

For z in $\mathfrak{g}_K$, let χ_z be the linear form $x \mapsto \langle z, x \rangle$ and let $\hat{I}_z$ be the ideal of Z_K generated by $T_1, \dots, T_\ell$ and the elements $x^p - x^{[p]} - \chi_z(x)^p$, for $x \in \mathfrak{g}_K$. Thus, I_{χ_z} is the image of $\hat{I}_z$ in $\mathrm{U}_{\chi_z}(\mathfrak{g}_K)$ by the quotient map $\mathrm{U}(\mathfrak{g}_K) \to \mathrm{U}_{\chi_z}(\mathfrak{g}_K)$. Consider on $\mathbb{N}^\ell$ the lexicographic order induced by the usual order of $\mathbb{N}$ and denote it by $\preccurlyeq$. For m a positive integer and for $\mathbf{i}$ in $\mathbb{N}^\ell$, denote by $\hat{I}_{z,m,\mathbf{i}}$ the ideal of Z_K generated by $\hat{I}_z^{m+1}$ and the elements $T^{\mathbf{j}}$ with $\mathbf{j}$ in $\mathbb{N}^\ell \smallsetminus \{\mathbf{i}\}$ such that $|\mathbf{j}| = m$ and $\mathbf{j} \preccurlyeq \mathbf{i}, \mathbf{j} \neq \mathbf{i}$.

Set:

$$\Lambda_{\ell,m} := \{\mathbf{i} \in \Lambda_\ell \;;\; |\mathbf{i}| = m\}.$$

In particular, $\Lambda_{\ell,m}$ is empty if $m > \ell(p-1)$.

Our basic reference concerning Azymaya algebras is [McR01, Chap. 13, §7]. What will be important for us is the following result, [McR01, Prop. 13.7.9]: *if A is an Azumaya algebra with center Z, then there is a one-to-one correspondence between the twosided ideals of A and the ideals of Z given by the maps $I \mapsto I \cap Z$, $J \mapsto JA$.*

Lemma 4.3. *Let z be a regular nilpotent element of $\mathfrak{g}_K$ and let m be a positive integer smaller than $\ell(p-1)$.*

(i) *The ideal $\hat{I}_z$ of Z_K is maximal and the localization at $\hat{I}_z$ of $\mathrm{U}(\mathfrak{g}_K)$ is an Azumaya algebra with center the localization of Z_K at $\hat{I}_z$.*

(ii) *The ideal $\mathrm{U}(\mathfrak{g}_K)\hat{I}_z$ of $\mathrm{U}(\mathfrak{g}_K)$ is maximal.*

(iii) *Let $\mathbf{i}$ be in $\Lambda_{\ell,m}$. Then $T^{\mathbf{i}}$ is not in $\mathrm{U}(\mathfrak{g}_K)\hat{I}_{z,m,\mathbf{i}}$.*

(iv) *For $a_{\mathbf{i}}$, $\mathbf{i} \in \Lambda_{\ell,m}$, in $\mathrm{U}(\mathfrak{g}_K)$, the following equivalence holds:*

$$\sum_{\mathbf{i} \in \Lambda_{\ell,m}} a_{\mathbf{i}} T^{\mathbf{i}} \in \mathrm{U}(\mathfrak{g}_K)\hat{I}_z^{m+1} \qquad \Longleftrightarrow \qquad \forall\, \mathbf{i} \in \Lambda_{\ell,m},\ a_{\mathbf{i}} \in \mathrm{U}(\mathfrak{g}_K)\hat{I}_z.$$

Proof. (i) To begin with, prove that $\hat{I}_z$ is the annihilator of χ_z in Z_K. Since z is nilpotent, χ_z vanishes $T_1, \dots, T_\ell$. Let $\{h_i, x_\alpha, i = 1, \dots, \ell,\ \alpha \in \mathcal{R}\}$ be a basis of $\mathfrak{g}_K$ derived from a Chevalley basis of $\mathfrak{g}$, where $\mathcal{R}$ is a root system of $\mathfrak{g}$. Since z is nilpotent, we can assume that z lies in the subalgebra generated by the positive vectors x_α of the above Chevalley basis. Hence, $\langle z, h_i\rangle = 0$ for $i = 1,\dots,\ell$. On the other hand, for $i \in \{1,\dots,\ell\}$, $h_i^{[p]} = h_i$ and for any $\alpha \in \mathcal{R}$, $x_\alpha^{[p]} = 0$ since $p > h$. Let $x \in \mathfrak{g}_K$ and write it as

$$x = \sum_{\alpha\in\mathcal{R}} a_\alpha x_\alpha + \sum_{i=1}^{\ell} a_i h_i, \qquad a_i,\, a_\alpha \in K.$$

Then

$$x^{[p]} = \sum_{\alpha} a_\alpha^p x_\alpha^{[p]} + \sum_{i=1}^{\ell} a_i^p h_i^{[p]} = \sum_{i=1}^{\ell} a_i^p h_i.$$

As a consequence, $\langle z\,, x^p - x^{[p]} - \chi_z(x)^p\rangle = 0$. This proves that $\hat{I}_z$ is contained in the annihilator of χ_z in Z_K. The other inclusion is clear from the equality (1). Hence $\hat{I}_z$ is a maximal ideal of Z_K. Since z is regular and since p is bigger than the Coxeter number of the root system of $\mathfrak{g}$, the localization of $\mathrm{U}(\mathfrak{g}_K)$ at $\hat{I}_z$ is an Azumaya algebra with center the localization of Z_K at $\hat{I}_z$; cf. [BG97, Thm. 4.10].

(ii) Denote by $\mathrm{U}(\mathfrak{g}_K)_z$ and $(Z_K)_z$ the localizations of $\mathrm{U}(\mathfrak{g}_K)$ and Z_K respectively at $\hat{I}_z$. By (i), $\mathrm{U}(\mathfrak{g}_K)_z$ is an Azumaya algebra with center $(Z_K)_z$. So, by [McR01, Prop. 13.7.9], for any ideal P of $\mathrm{U}(\mathfrak{g}_K)_z$, P is the ideal generated by $P \cap (Z_K)_z$. Then $\mathrm{U}(\mathfrak{g}_K)\hat{I}_z$ is a maximal ideal of $\mathrm{U}(\mathfrak{g}_K)$ since $K + \hat{I}_z = Z_K$.

(iii) Let $\mathbf{i}$ be in $\Lambda_{\ell,m}$ and suppose that $T^{\mathbf{i}}$ is in $\mathrm{U}(\mathfrak{g}_K)\hat{I}_{z,m,\mathbf{i}}$. A contradiction is expected. By (i) and [McR01, Prop. 13.7.9], $\hat{I}_{z,m,\mathbf{i}} = Z_K \cap \mathrm{U}(\mathfrak{g}_K)\hat{I}_{z,m,\mathbf{i}}$ since $K + \hat{I}_z = Z_K$. Hence $T^{\mathbf{i}}$ is in $\hat{I}_{z,m,\mathbf{i}}$. Then the contradiction follows from [V72, Thm. 3.1].

(iv) The converse implication is clear. Let us prove the direct implication. Let $a_{\mathbf{i}}$, $\mathbf{i} \in \Lambda_{\ell,m}$, be in $\mathrm{U}(\mathfrak{g}_K)$ such that

$$\sum_{\mathbf{i}\in\Lambda_{\ell,m}} a_{\mathbf{i}} T^{\mathbf{i}} \in \mathrm{U}(\mathfrak{g}_K)\hat{I}_z^{m+1}.$$

Suppose that the $a_{\mathbf{i}}$'s are not all in $\mathrm{U}(\mathfrak{g}_K)\hat{I}_z$. A contradiction is expected. Let $\mathbf{i}$ be the biggest element of $\Lambda_{\ell,m}$ such that $a_{\mathbf{i}}$ is not in $\mathrm{U}(\mathfrak{g}_K)\hat{I}_z$. Then $a_{\mathbf{i}}T^{\mathbf{i}}$ is in $\mathrm{U}(\mathfrak{g}_K)\hat{I}_{z,m,\mathbf{i}}$. Since $T^{\mathbf{i}}$ is in the center of $\mathrm{U}(\mathfrak{g}_K)$, the subset of elements a of $\mathrm{U}(\mathfrak{g}_K)$ such that $aT^{\mathbf{i}}$ is in $\mathrm{U}(\mathfrak{g}_K)\hat{I}_{z,m,\mathbf{i}}$ is an ideal containing $\mathrm{U}(\mathfrak{g}_K)\hat{I}_z$. By (iii), this ideal is strictly contained in $\mathrm{U}(\mathfrak{g}_K)$. So it equals $\mathrm{U}(\mathfrak{g}_K)\hat{I}_z$ by (ii), whence the contradiction. □

Proposition 4.4. *Let χ be in $\mathfrak{g}_K^*$ and let m be a positive integer. Then the canonical morphism*

$$\mathrm{U}_\chi(\mathfrak{g}_K) \otimes_K I_\chi^m \longrightarrow \mathrm{U}_\chi(\mathfrak{g}_K)I_\chi^m$$

defines through the quotients an isomorphism

$$\mathrm{U}_\chi(\mathfrak{g}_K)/\mathrm{U}_\chi(\mathfrak{g}_K)I_\chi \otimes_K I_\chi^m/I_\chi^{m+1} \longrightarrow \mathrm{U}_\chi(\mathfrak{g}_K)I_\chi^m/\mathrm{U}_\chi(\mathfrak{g}_K)I_\chi^{m+1}.$$

Proof. Since $\mathrm{U}_\chi(\mathfrak{g}_K)I_\chi^m/\mathrm{U}_\chi(\mathfrak{g}_K)I_\chi^{m+1}$ is a quotient of $\mathrm{U}_\chi(\mathfrak{g}_K)I_\chi^m$, there is a canonical morphism

$$\mathrm{U}_\chi(\mathfrak{g}_K) \otimes_K I_\chi^m \longrightarrow \mathrm{U}_\chi(\mathfrak{g}_K)I_\chi^m/\mathrm{U}_\chi(\mathfrak{g}_K)I_\chi^{m+1}.$$

Moreover, this morphism is surjective. Then it defines through the quotient a surjective morphism

$$\mathrm{U}_\chi(\mathfrak{g}_K) \otimes_K I_\chi^m/I_\chi^{m+1} \longrightarrow \mathrm{U}_\chi(\mathfrak{g}_K)I_\chi^m/\mathrm{U}_\chi(\mathfrak{g}_K)I_\chi^{m+1}$$

and this morphism defines through the quotient a surjective morphism

$$\mathrm{U}_\chi(\mathfrak{g}_K)/\mathrm{U}_\chi(\mathfrak{g}_K)I_\chi \otimes_K I_\chi^m/I_\chi^{m+1} \longrightarrow \mathrm{U}_\chi(\mathfrak{g}_K)I_\chi^m/\mathrm{U}_\chi(\mathfrak{g}_K)I_\chi^{m+1}$$

Since it is a morphism of finite dimensionnal K-vector spaces, it suffices to prove that these two spaces have the same dimension. By Lemma 4.2, it suffices to find some χ such that this morphism is an isomorphism.

By Lemma 4.3,(iv), if z is a regular nilpotent element of $\mathfrak{g}_K$, then the kernel of the morphism

$$\mathrm{U}_{\chi_z}(\mathfrak{g}_K) \otimes_K I_{\chi_z}^m \longrightarrow \mathrm{U}_{\chi_z}(\mathfrak{g}_K)I_{\chi_z}^m/\mathrm{U}_{\chi_z}(\mathfrak{g}_K)I_{\chi_z}^{m+1}$$

equals $\mathrm{U}_{\chi_z}(\mathfrak{g}_K)I_{\chi_z} \otimes_K I_{\chi_z}^m$ so that the morphism

$$\mathrm{U}_{\chi_z}(\mathfrak{g}_K)/\mathrm{U}_{\chi_z}(\mathfrak{g}_K)I_{\chi_z} \otimes_K I_{\chi_z}^m/I_{\chi_z}^{m+1} \longrightarrow \mathrm{U}_{\chi_z}(\mathfrak{g}_K)I_{\chi_z}^m/\mathrm{U}_{\chi_z}(\mathfrak{g}_K)I_{\chi_z}^{m+1}$$

is an isomorphism, whence the proposition. □

Recall that χ_e is the linear form $x \mapsto \langle e\,,x\rangle$. Set

$$Z_{K,e} := Z_{K,\chi_e} \quad \text{ and } \quad I_e := I_{\chi_e}.$$

By [V72, Theorem 3.1] and [Pr02, Theorem 2.3, (ii)], the restriction to $Z_{K,e}$ of the representation $\mathrm{U}_e(\mathfrak{g}_K) \to H$ is an embedding. Identify $Z_{K,e}$ with a subalgebra of H through this representation.

Corollary 4.5. (i) *For m positive integer, the canonical morphism*

$$H \otimes_K I_e^m \longrightarrow HI_e^m$$

defines through the quotients an isomorphism

$$H/HI_e \otimes_K I_e^m/I_e^{m+1} \longrightarrow HI_e^m/HI_e^{m+1}.$$

(ii) *For some K-subspace E of H, the linear map*

$$E \otimes_K Z_{K,e} \longrightarrow H, \qquad v{\otimes}a \longmapsto va$$

is an isomorphism of K-spaces.

Proof. (i) By [Pr02, Thm. 2.3, Thm. 2.4 and Prop. 2.6],

$$\mathrm{U}_e(\mathfrak{g}_K) = \mathrm{Mat}_d(H) \quad \text{ with } \quad d = p^{\frac{1}{2}\dim G_K.e}.$$

Moreover, since $p > h$, $Z_{K,e}$ is the center of $\mathrm{U}_e(\mathfrak{g}_K)$ so that $Z_{K,e}$ is the center of H. Let $a_{\mathbf{i}}$, $\mathbf{i} \in \Lambda_{\ell,m}$, be in H such that

$$\sum_{\mathbf{i}\in\Lambda_{\ell,m}} a_{\mathbf{i}}T^{\mathbf{i}} \in HI_e^{m+1}.$$

It results from Proposition 4.4 with $\chi = \chi_e$ that the $a_{\mathbf{i}}$'s are all in $\mathrm{U}_e(\mathfrak{g}_K)I_e$. Then, since

$$\mathrm{Mat}_d(H)I_e \cap H = HI_e,$$

the $a_{\mathbf{i}}$'s are all in HI_e. Therefore, the canonical morphism

$$H/HI_e \otimes_K I_e^m/I_e^{m+1} \longrightarrow HI_e^m/HI_e^{m+1}$$

is injective. But this morphism is surjective by definition. This concludes the proof.

(ii) Let E be a K-subspace of H such that the restriction to E of the quotient morphism $H \to H/HI_e$ is an isomorphism and denote by Θ the linear map

$$E \otimes_K Z_{K,e} \longrightarrow H, \qquad v{\otimes}a \longmapsto va.$$

By (i) with $m = 0$, Θ is injective and, again by (i), for all m,

$$H \subset \Theta(E \otimes_K Z_{K,e}) + I_e^m.$$

The assertion follows since $I_e^m = \{0\}$ for $m > \ell(p-1)$. □

4.4. Let $\mathrm{S}_e(\mathfrak{g}_K^e)$ be the quotient of the symmetric algebra $\mathrm{S}(\mathfrak{g}_K^e)$ by the ideal generated by the elements x^p, with $x \in \mathfrak{g}_K^e$, and let $\mathrm{U}_e(\mathfrak{g}_K^e)$ be the quotient of the enveloping algebra $\mathrm{U}(\mathfrak{g}_K^e)$ by the ideal generated by the elements $x^p - x^{[p]}$, with x in $\mathfrak{g}_K^e$. Since e is orthogonal to $\mathfrak{g}_K^e$, the canonical injection from $\mathrm{U}(\mathfrak{g}_K^e)$ into $\mathrm{U}(\mathfrak{g}_K)$ induces an embedding of $\mathrm{U}_e(\mathfrak{g}_K^e)$ into $\mathrm{U}_e(\mathfrak{g}_K)$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, let $\mathcal{R}$ be the root system of $(\mathfrak{g}, \mathfrak{h})$ and let $W(\mathcal{R})$ be the corresponding Weyl group. Let $\mathfrak{h}_{\mathbb{Z}}$ be the sub-$\mathbb{Z}$-module of $\mathfrak{h}$ generated by the coroots of $\mathcal{R}$, and set:

$$\mathfrak{h}_N := \mathbb{Z}[1/N!] \otimes_{\mathbb{Z}} \mathfrak{h}_{\mathbb{Z}}, \qquad \mathfrak{h}_K := K \otimes_{\mathbb{Z}[1/N!]} \mathfrak{h}_N.$$

Since $\langle .\,,.\rangle$ is nondegenerate, the duals $\mathfrak{g}_K^*$ and $\mathfrak{h}_K^*$ of $\mathfrak{g}_K$ and $\mathfrak{h}_K$ respectively identify with $\mathfrak{g}_K$ and $\mathfrak{h}_K$ respectively so that $\mathrm{S}(\mathfrak{g}_K)$ and $\mathrm{S}(\mathfrak{h}_K)$ are the algebras of polynomial functions on $\mathfrak{g}_K$ and $\mathfrak{h}_K$ respectively. The Weyl group $W(\mathcal{R})$ defines through the quotient an action on $\mathfrak{h}_K$. Since $p > h$, $W(\mathcal{R})$ is embedded in $\mathrm{GL}(\mathfrak{h}_K)$. By [V72, Prop. 2.1], there exists an isomorphism δ from $\mathrm{U}(\mathfrak{g}_K)^{G_K}$ onto $\mathrm{S}(\mathfrak{h}_K)^{W(\mathcal{R})}$. Moreover, $\mathrm{U}(\mathfrak{g}_K)^{G_K}$ is a polynomial algebra generated by $T_1, \ldots, T_\ell$. By [SS70, §3.17], the restriction map from $\mathfrak{g}_K$ to $\mathfrak{h}_K$ induces an isomorphism from $\mathrm{S}(\mathfrak{g}_K)^{G_K}$ onto $\mathrm{S}(\mathfrak{h}_K)^{W(\mathcal{R})}$. For $i \in \{1, \ldots, \ell\}$, let S_i be the element of $\mathrm{S}(\mathfrak{g}_K)^{G_K}$ such that $\delta(T_i)$ is the restriction of S_i to $\mathfrak{h}_K$.

Since the restriction of $\langle .\,,.\rangle$ to $\mathfrak{g}_K^e \times \mathfrak{g}_K^f$ is nondegenerate, $\mathfrak{g}_K^f$ identifies with the dual of $\mathfrak{g}_K^e$ and $\Bbbk[e + \mathfrak{g}_K^f]$ identifies with $\mathrm{S}(\mathfrak{g}_K^e)$. For $i = 1, \ldots, \ell$, let S'_i be the image in $\mathrm{S}_e(\mathfrak{g}_K^e)$ of the restriction of S_i to $e + \mathfrak{g}_K^f$.

Proposition 4.6. *There is an isomorphism*

$$\tau \;:\; H \;\longrightarrow\; \mathrm{S}_e(\mathfrak{g}_K^e)$$

from the K-space H onto the K-space $\mathrm{S}_e(\mathfrak{g}_K^e)$ such that $\tau(Z_{K,e})$ is the subalgebra of $\mathrm{S}_e(\mathfrak{g}_K^e)$ generated by $S'_1, \ldots, S'_\ell$ and such that $\tau(ab) = \tau(a)\tau(b)$ for all (a, b) in $H \times Z_{K,e}$.

Proof. Recall that $x_1, \ldots, x_m$ is the basis of $\mathfrak{p}_{K,+}$ introduced in Subsection 4.1. Order it so that $x_1, \ldots, x_r$ is a basis of $\mathfrak{g}_K^e$. For θ in H, denote by $\overline{\theta}$ its image in $\mathrm{gr}(H)$ by the canonical map. By [Pr02, Thm. 3.4], there exist $\theta_1, \ldots, \theta_r$ in H such that the monomials $\overline{\theta}_1^{a_1} \cdots \overline{\theta}_r^{a_r}$ and $\theta_1^{a_1} \cdots \theta_r^{a_r}$, with $0 \leqslant a_i \leqslant p-1$, form bases of $\mathrm{gr}(H)$ and H respectively. Moreover, there exists an isomorphism from the K-algebra $\mathrm{gr}(H)$ onto the K-algebra $\mathrm{S}_e(\mathfrak{g}_K^e)$ such that $x_1, \ldots, x_r$ is the image of $\theta_1, \ldots, \theta_r$ respectively. Let τ be the linear isomorphism from H onto $\mathrm{S}_e(\mathfrak{g}_K^e)$ such that

$$\tau(\theta_1^{a_1} \cdots \theta_r^{a_r}) = x_1^{a_1} \cdots x_r^{a_r}$$

for all $(a_1, \ldots, a_r) \in \Lambda_r$. It remains to prove that for $i \in \{1, \ldots, \ell\}$ and for a in H, $\tau(aT_i) = \tau(a)S'_i$.

Let A be the subspace of $\mathrm{U}(\mathfrak{g}_K)$ generated by the monomials $x_{r+1}^{a_{r+1}} \cdots x_m^{a_m}$, with $(a_{r+1}, \ldots, a_m) \in \mathbb{N}^{m-r} \smallsetminus \{0\}$, and let $\mathfrak{m}'_K$ be the orthogonal complement to e in $\mathfrak{m}_K$. By the PBW theorem,

$$T_i - \sum_{k \in \mathbb{N}} \sum_{\mathbf{j} \in \mathbb{N}^r} c_{i,\mathbf{j},k} x^{\mathbf{j}} f^k \in A + \mathrm{U}(\mathfrak{g}_K)\mathfrak{m}'_K \tag{2}$$

with the $c_{i,\mathbf{j},k}$'s in K. By [Pr02, Thm. 3.4], $\tau(T_i)$ is the polynomial function on $\mathfrak{g}_K^f$,

$$v \longmapsto \sum_{k \in \mathbb{N}} \sum_{\mathbf{j} \in \mathbb{N}^r} c_{i,\mathbf{j},k} \langle v\,, x_1\rangle^{j_1} \cdots \langle v\,, x_r\rangle^{j_r} \langle e\,, f\rangle^k$$

By definition, S_i is the G_K-invariant polynomial function on $\mathfrak{g}_K$ such that its restriction to $\mathfrak{h}_K$ equals $\delta(T_i)$. Moreover, since $p > h$, S_i is the image of T_i in $\mathrm{S}(\mathfrak{g}_K)$; to see that, we follow the proof of [Di74, Thm. 7.4.5]. Hence

$$S_i - \sum_{k \in \mathbb{N}} \sum_{\mathbf{j} \in \mathbb{N}^r} c_{i,\mathbf{j},k} x^{\mathbf{j}} f^k \in \sum_{l=r+1}^{m} \mathrm{S}(\mathfrak{g}_K)x_l + \mathrm{S}(\mathfrak{g}_K)\mathfrak{m}'_K$$

As a result, for v in $\mathfrak{g}_K^f$,

$$S_i(e+v) = \sum_{k\in\mathbb{N}} \sum_{\mathbf{j}\in\mathbb{N}^r} c_{i,\mathbf{j},k} \langle v\,,x_1\rangle^{j_1} \cdots \langle v\,,x_r\rangle^{j_r} \langle e\,,f\rangle^k$$

so that $S'_i = \tau(T_i)$.

Let a be in H. By [Pr02, Thm. 3.4],

$$a - \sum_{k\in\mathbb{N}} \sum_{\mathbf{j}\in\mathbb{N}^r} \gamma_{a,\mathbf{j},k} x^{\mathbf{j}} f^k \in A + \mathrm{U}(\mathfrak{g}_K)\mathfrak{m}'_K \tag{3}$$

with the $\gamma_{a,\mathbf{j},k}$'s in K, and $\tau(a)$ is the polynomial function on $\mathfrak{g}^f$,

$$v \longmapsto \sum_{k\in\mathbb{N}} \sum_{\mathbf{j}\in\mathbb{N}^r} \gamma_{a,\mathbf{j},k} \langle v\,,x_1\rangle^{j_1} \cdots \langle v\,,x_r\rangle^{j_r} \langle e\,,f\rangle^k.$$

From the equalities (2) and (3), it results that

$$\begin{array}{rcl} aT_i - \displaystyle\sum_{k\in\mathbb{N}} \sum_{\mathbf{j}\in\mathbb{N}^r} \gamma_{a,\mathbf{j},k} x^{\mathbf{j}} T_i f^k & \in & AT_i + \mathrm{U}(\mathfrak{g}_K)\mathfrak{m}'_K \\ AT_i & \subset & \displaystyle\sum_{k\in\mathbb{N}} c_{i,\mathbf{j},k} A x^{\mathbf{j}} f^k + AA + \mathrm{U}(\mathfrak{g}_K)\mathfrak{m}'_K \\ x^{\mathbf{j}} T_i f^k - \displaystyle\sum_{k'\in\mathbb{N}} \sum_{\mathbf{j}'\in\mathbb{N}^r} c_{i,\mathbf{j}',k'} x^{\mathbf{j}} x^{\mathbf{j}'} f^{k+k'} & \subset & x^{\mathbf{j}} A f^k + \mathrm{U}(\mathfrak{g}_K)\mathfrak{m}'_K. \end{array}$$

For a_1, a_2 in A of filtrations degree $|a_1|_e$ and $|a_2|_e$ respectively, $a_1a_2\otimes 1 = a_3\otimes 1 + a_4\otimes 1$ where a_3 is in A and a_4 is a linear combination of the $x^{\mathbf{i}}z^{\mathbf{j}}\otimes 1$'s, with $|(\mathbf{i},\mathbf{j})|_e$ smaller than $|a_1|_e + |a_2|_e$. Moreover, for $\mathbf{j}$ in $\mathbb{N}^r$, $a_1x^{\mathbf{j}}\otimes 1 = x^{\mathbf{j}}a_1\otimes 1 + a_5\otimes 1$ where a_5 is a linear combination of the $x^{\mathbf{i}}z^{\mathbf{k}}\otimes 1$'s, with $|(\mathbf{i},\mathbf{k})|_e$ smaller than $|a_1|_e + |(\mathbf{j},\mathbf{0})|_e$. At last, $(x^{\mathbf{j}}x^{\mathbf{j}'} - x^{\mathbf{j}+\mathbf{j}'}\otimes 1)$ is a linear combination of the $x^{\mathbf{k}}\otimes 1$'s with $|(\mathbf{k},\mathbf{0})|_e$ smaller than $|(\mathbf{j},\mathbf{0})|_e + |(\mathbf{j}',\mathbf{0})|_e$. All this shows that $\tau(aT_i)$ is the polynomial function on $\mathfrak{g}^f$

$$v \longmapsto \sum_{(k,k')\in\mathbb{N}^2} \sum_{(\mathbf{j},\mathbf{j}')\in\mathbb{N}^r\times\mathbb{N}^r} c_{i,\mathbf{j},k}\gamma_{a,\mathbf{j}',k'} \langle v\,,x_1\rangle^{j_1+j'_1} \cdots \langle v\,,x_r\rangle^{j_r+j'_r} \langle e\,,f\rangle^{k+k'},$$

whence $\tau(aT_i) = \tau(a)\tau(T_i)$. □

Henceforth, E is a subspace of H such that the linear map $E\otimes_K Z_{K,e} \longrightarrow H$, $v\otimes a \longmapsto va$ is an isomorphism of K-spaces. The existence of such a subspace is provided by Corollary 4.5,(ii).

Corollary 4.7. *The morphism*

$$\tau(E)\otimes_K \tau(Z_{K,e}) \longrightarrow \mathrm{S}_e(\mathfrak{g}_K^e), \qquad v\otimes a \longmapsto va$$

is an isomorphism of K-spaces.

Proof. By Proposition 4.6, $\tau(E)\tau(Z_{K,e}) = \mathrm{S}_e(\mathfrak{g}_K^e)$. In particular, the K-linear map

$$\tau(E)\otimes_K \tau(Z_{K,e}) \longrightarrow \mathrm{S}_e(\mathfrak{g}_K^e), \qquad v\otimes a \longmapsto va$$

is surjective. Since the K-spaces $E\otimes_K Z_{K,e}$, H, $\tau(E)\otimes_K\tau(Z_{K,e})$ and $\tau(H)$ are finite dimensional of the same dimension, this map is an isomorphism. □

4.5. Let $\mathbf{m}$ be the image in $\mathrm{S}_e(\mathfrak{g}_K^e)$ of the augmentation ideal of $\mathrm{S}(\mathfrak{g}_K^e)$ by the quotient map. Then the sequence $\mathbf{m}, \mathbf{m}^2, \ldots$ is a decreasing filtration of $\mathrm{S}_e(\mathfrak{g}_K^e)$ such that the graded space associated with this filtration is the algebra $\mathrm{S}_e(\mathfrak{g}_K^e)$. This filtration induces a filtration on $\tau(Z_{K,e})$ and the graded algebra associated with this filtration is a subalgebra of $\mathrm{S}_e(\mathfrak{g}_K^e)$ denoted by $\mathrm{gr}(\tau(Z_{K,e}))$.

Proposition 4.8. *The linear map*

$$\mathrm{gr}(\tau(E)) \otimes_K \mathrm{gr}(\tau(Z_{K,e})) \longrightarrow \mathrm{S}_e(\mathfrak{g}_K^e), \qquad v \otimes a \longmapsto va$$

is an isomorphism.

Proof. By Corollary 4.7, the linear map

$$\tau(E) \otimes_K \tau(Z_{K,e}) \longrightarrow \mathrm{S}_e(\mathfrak{g}_K^e), \qquad v \otimes a \longmapsto va$$

is an isomorphism. The filtration on $\mathrm{S}_e(\mathfrak{g}_K^e)$ induces a filtration on $\tau(E)$ and the graded space $\mathrm{gr}(\tau(E))$ associated with this filtration is a subspace of $\mathrm{S}_e(\mathfrak{g}_K^e)$ of the same dimension. For d nonnegative integer, denote by $\mathrm{gr}(\tau(E))_d$ the subspace of degree d of $\mathrm{gr}(\tau(E))$ and set:

$$\mathrm{gr}(\tau(E))^{(d)} := \bigoplus_{i \leqslant d} \mathrm{gr}(\tau(E))_i.$$

Let $\mathrm{gr}(\tau(Z_{K,e}))_+$ be the augmentation ideal of $\mathrm{gr}(\tau(Z_{K,e}))$ and prove by induction on d that

$$\mathrm{S}_e(\mathfrak{g}_K^e) \subset \mathrm{gr}(\tau(E)) + \mathrm{S}_e(\mathfrak{g}_K^e)\mathrm{gr}(\tau(Z_{K,e}))_+ + \mathbf{m}^{d+1}.$$

Since $\mathrm{gr}(\tau(E))^{(0)} = K$, the inclusion is clear for $d = 0$. Suppose that it is true for any integer smaller than $d-1$ and prove the inclusion for d. By induction hypothesis, it suffices to prove that for a homogeneous polynomial a of degree d in $\mathrm{S}_e(\mathfrak{g}_K^e)$,

$$a \in \mathrm{gr}(E) + \mathrm{S}_e(\mathfrak{g}_K^e)\mathrm{gr}(\tau(Z_{K,e}))_+ + \mathbf{m}^{d+1}.$$

Let a be a homogeneous polynomial of degree d in $\mathrm{S}_e(\mathfrak{g}_K^e)$, and let $\{v_1, \ldots, v_m\}$ be a basis of E such that its image in $\mathrm{gr}(E)$ is linearly free. Then

$$a = \sum_{i=1}^{m} v_i a_i$$

for some $a_1, \ldots, a_m$ in $\tau(Z_{K,e})$ and,

$$a \in \sum_{i \in I_d} v_i a_i + \mathbf{m}^{d+1}$$

where I_d is the subset of $i \in \{1, \ldots, m\}$ such that v_i is in $\mathrm{S}_e(\mathfrak{g}_K^e) \setminus \mathbf{m}^{d+1}$. For i in I_d such that v_i is not in $\mathbf{m}^d$, a_i is in $\mathbf{m}$ so that its image in $\mathrm{gr}(\tau(Z_{K,e}))$ is in $\mathrm{gr}(\tau(Z_{K,e}))_+$. As a result,

$$a \in \mathrm{gr}(E) + \mathrm{S}_e(\mathfrak{g}_K^e)\mathrm{gr}(\tau(Z_{K,e}))_+ + \mathbf{m}^{d+1}.$$

Since $\mathbf{m}^d = \{0\}$ for d sufficiently big, one deduces that

$$\mathrm{S}_e(\mathfrak{g}_K^e) \subset \mathrm{gr}(\tau(E)) + \mathrm{S}_e(\mathfrak{g}_K^e)\mathrm{gr}(\tau(Z_{K,e}))_+.$$

Then, by induction on i, one gets

$$\mathrm{S}_e(\mathfrak{g}_K^e) \subset \mathrm{gr}(\tau(E))\mathrm{S}_e(\mathfrak{g}_K^e) + \mathrm{S}_e(\mathfrak{g}_K^e)\mathrm{gr}(\tau(Z_{K,e}))^i.$$

For i sufficiently big, $\mathrm{gr}(\tau(Z_{K,e}))^i = \{0\}$. Therefore, the linear map

$$\mathrm{gr}(\tau(E)) \otimes_K \mathrm{gr}(\tau(Z_{K,e})) \longrightarrow \mathrm{S}_e(\mathfrak{g}_K^e), \qquad v \otimes a \longmapsto va$$

is surjective. As the K-spaces $\tau(E) \otimes_K \mathrm{gr}(\tau(Z_{K,e}))$, $\mathrm{gr}(\tau(E)) \otimes_K \mathrm{gr}(\tau(Z_{K,e}))$ and $\mathrm{S}_e(\mathfrak{g}_K^e)$ are finite dimensional of the same dimension, this map is an isomorphism. □

For q in $\mathrm{S}(\mathfrak{g}_K)$, denote by ${}^e q$ the initial component of the restriction of q to $e + \mathfrak{g}_K^f$.

Corollary 4.9. *Let $q_1, \ldots, q_\ell$ be homogeneous generators of $\mathrm{S}(\mathfrak{g}_K)^{G_K}$ such that ${}^e q_1, \ldots, {}^e q_\ell$ are algebraically independent over K.*

(i) *The set*

$$\{{}^e q_1^{i_1} \cdots {}^e q_\ell^{i_\ell},\ \mathbf{i} = (i_1, \ldots, i_\ell) \in \Lambda_\ell\}$$

is a basis of the K-space $\mathrm{gr}(\tau(Z_{k,e}))$.

(ii) *For $a_1, \ldots, a_\ell$ in $\mathrm{S}_e(\mathfrak{g}_K^e)$, if*

$$a_1({}^e q_1) + \cdots + a_\ell({}^e q_\ell) = 0$$

then $a_1, \ldots, a_\ell$ are linear combinations with coefficients in $\mathrm{S}_e(\mathfrak{g}_K^e)$ of ${}^e q_1, \ldots, {}^e q_\ell$.

Proof. (i) Since ${}^e q_1, \ldots, {}^e q_\ell$ are algebraically independent over K, the jacobian matrix

$$(\frac{\partial({}^e q_i)}{\partial x_j},\ 1 \leqslant i \leqslant \ell, 1 \leqslant j \leqslant r)$$

has rank ℓ. This means that in $K(x_1, \ldots, x_r)$, the quotient field of $\mathrm{S}(\mathfrak{g}_K^e)$, the elements ${}^e q_1, \ldots, {}^e q_\ell$ are p-independent. Hence the sequence

$${}^e q_1^{i_1} \cdots {}^e q_\ell^{i_\ell}, \quad \mathbf{i} = (i_1, \ldots, i_\ell) \in \Lambda_\ell,$$

of elements of $\mathrm{S}_e(\mathfrak{g}_K^e)$ is linearly free over K. Since $q_1, \ldots, q_\ell$ are homogeneous generators of $\mathrm{S}(\mathfrak{g}_K)^{G_K}$, the algebra $\tau(Z_{K,e})$ is generated by the restrictions of $q_1, \ldots, q_\ell$ to $e + \mathfrak{g}_K^f$, [SS70, §3.17]. So, for a in $\tau(Z_{K,e})$, a is the restriction to $e + \mathfrak{g}_K^f$ of

$$\sum_{\mathbf{i}=(i_1,\ldots,i_\ell)\in\Lambda_\ell} c_{\mathbf{i}} q_1^{i_1} \cdots q_\ell^{i_\ell}$$

for some $c_{\mathbf{i}}, \mathbf{i} \in \Lambda_\ell$ in K so that the image $\overline{a}$ of a in $\mathrm{gr}(\tau(Z_{K,e}))$ equals

$$\sum_{\mathbf{i}=(i_1,\ldots,\, i_\ell)\in\Lambda_\ell} c'_{\mathbf{i}}\ {}^e q_1^{i_1} \cdots {}^e q_\ell^{i_\ell}$$

where $c'_{\mathbf{i}} = c_{\mathbf{i}}$ if $\overline{a}$ and ${}^e q_1^{i_1} \cdots {}^e q_\ell^{i_\ell}$ have the same degree, and $c'_{\mathbf{i}} = 0$ otherwise.

(ii) Let $a_1, \ldots, a_\ell$ be in $\mathrm{S}_e(\mathfrak{g}_K^e)$ such that

$$a_1({}^e q_1) + \cdots + a_\ell({}^e q_\ell) = 0.$$

Let $v_1, \ldots, v_m$ be a basis of $\mathrm{gr}(\tau(E))$. By (i) and Proposition 4.8, for $i = 1, \ldots, \ell$,

$$a_i = \sum_{j=1}^{m} \sum_{\mathbf{k}\in\Lambda_\ell} c_{i,j,\mathbf{k}}\, v_j\, {}^e q_1^{k_1} \cdots {}^e q_\ell^{k_\ell}$$

with the $c_{i,j,\mathbf{k}}$'s in K. As a result,

$$\sum_{j=1}^{m} v_j \otimes (\sum_{i=1}^{\ell} \sum_{\mathbf{k}\in\Lambda_\ell} c_{i,j,\mathbf{k}}({}^e q_i\ {}^e q_1^{k_1} \cdots {}^e q_\ell^{k_\ell}) = 0$$

so that

$$\sum_{i=1}^{\ell} \sum_{\mathbf{k}\in\Lambda_\ell} c_{i,j,\mathbf{k}}({}^e q_i\ {}^e q_1^{k_1} \cdots {}^e q_\ell^{k_\ell}) = 0$$

for $j = 1, \ldots, m$. By (i), it follows that $c_{i,j,\mathbf{0}} = 0$ for all (i, j), whence the statement. □

4.6. **Proof of Theorem 4.1.** Let $\mathfrak{g}_{\mathbb{Z}}$ be a $\mathbb{Z}$-form of $\mathfrak{g}$ such that (e, h, f) is an $\mathfrak{sl}_2$-triple of the $\mathbb{Q}$-form of $\mathfrak{g}_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} \mathfrak{g}_{\mathbb{Z}}$ of $\mathfrak{g}$. Let us suppose that for some generators $q_1, \dots, q_\ell$ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent over $\Bbbk$. We aim to prove that e is good. First of all, since $\mathfrak{g}_{\mathbb{Q}}$ is a $\mathbb{Q}$-form of $\mathfrak{g}$ containing e, h, f, there exist homogeneous generators $q'_1, \dots, q'_\ell$ of $\mathrm{S}(\mathfrak{g}_{\mathbb{Q}})^{\mathfrak{g}_{\mathbb{Q}}}$ such that ${}^e q'_1, \dots, {}^e q'_\ell$ are algebraically independent over $\mathbb{Q}$. So, one can suppose that $q_1, \dots, q_\ell$ are in $\mathrm{S}(\mathfrak{g}_{\mathbb{Q}})^{\mathfrak{g}_{\mathbb{Q}}}$.

Let $\{y_1, \dots, y_r\}$ be a basis of $\mathfrak{g}^f_{\mathbb{Q}}$ and let $\{x_1, \dots, x_n\}$ be a basis of $\mathfrak{g}_{\mathbb{Z}}$. By the hypothesis, for some $(v_1, \dots, v_r)$ in $\mathbb{Z}^r$ and $(u_1, \dots, u_n)$ in $\mathbb{Z}^n$, the value at $v_1 y_1 + \cdots + v_r y_r$ of a ℓ-order minor of the jacobian matrix

$$(\frac{\partial({}^e q_i)}{\partial y_j},\ 1 \leqslant i \leqslant \ell, 1 \leqslant j \leqslant r)$$

is a rational number c_0 different from 0, and the value at $u_1 x_1 + \cdots + u_n x_n$ of a ℓ-order minor of the jacobian matrix

$$(\frac{\partial q_i}{\partial x_j},\ 1 \leqslant i \leqslant \ell, 1 \leqslant j \leqslant n)$$

is a rational number $c_{0,0}$ different from 0.

Let $d_1, \dots, d_\ell$ be the degrees of ${}^e q_1, \dots, {}^e q_\ell$ respectively, and denote by J the ideal of $\mathrm{S}(\mathfrak{g}^e_{\mathbb{Q}})$ generated by ${}^e q_1, \dots, {}^e q_\ell$. For d positive integer, denote by $\mathrm{S}_d(\mathfrak{g}^e_{\mathbb{Q}})$ and J_d the subspace of homogeneous elements of degree d of $\mathrm{S}(\mathfrak{g}^e_{\mathbb{Q}})$ and J respectively. Suppose that for some positive integer d there exist homogeneous elements $a_1, \dots, a_\ell$ of degrees $d - d_1, \dots, d - d_\ell$ respectively, not all in J, such that

$$a_1({}^e q_1) + \cdots + a_\ell({}^e q_\ell) = 0.$$

A contradiction is expected. Then for some μ in the orthogonal complement of $J_{d-d_1} \times \cdots \times J_{d-d_\ell}$ in the dual of $\mathrm{S}_{d-d_1}(\mathfrak{g}^e_{\mathbb{Q}}) \times \cdots \times \mathrm{S}_{d-d_\ell}(\mathfrak{g}^e_{\mathbb{Q}})$, $c_1 := \mu(a_1, \dots, a_\ell)$ is a rational number different from zero.

Let N be a sufficiently big positive integer verifying the conditions of Subsection 4.2 and the following conditions, where $\mathfrak{g}_N := \mathbb{Z}[1/N!] \otimes_{\mathbb{Z}} \mathfrak{g}_{\mathbb{Z}}$:

1) c_0, $c_{0,0}$, c_1 are invertible elements of $\mathbb{Z}[1/N!]$,
2) $q_1, \dots, q_\ell$ are in $\mathrm{S}(\mathfrak{g}_N)$,
3) $y_1, \dots, y_r$ are in $\mathfrak{g}^f_N$,
4) $a_1, \dots, a_\ell$ are in $\mathrm{S}(\mathfrak{g}^e_N)$,
5) μ is the extension to $\mathrm{S}_{d-d_1}(\mathfrak{g}^e_{\mathbb{Q}}) \times \cdots \times \mathrm{S}_{d-d_\ell}(\mathfrak{g}^e_{\mathbb{Q}})$ of a linear form μ_0 on the $\mathbb{Z}[1/N!]$-module $\mathrm{S}_{d-d_1}(\mathfrak{g}^e_N) \times \cdots \times \mathrm{S}_{d-d_\ell}(\mathfrak{g}^e_N)$.

Let p be a positive integer bigger than N and d. Let $\mathfrak{M}_p$ be a maximal ideal of $\mathbb{Z}[1/N!]$ containing p, let K be an algebraic closure of $\mathbb{Z}[1/N!]/\mathfrak{M}_p$ and set:

$$\mathfrak{g}_K := K \otimes_{\mathbb{Z}[N!]} \mathfrak{g}_N.$$

Let G_K be a simple, simply connected algebraic K-group such that $\mathfrak{g}_K = \mathrm{Lie}(G_K)$. Because of the above conditions, the above data reduce modulo $\mathfrak{M}_p$. For a in $\mathrm{S}(\mathfrak{g}_N)$, denote again by a the element $1 \otimes a$ of $\mathrm{S}(\mathfrak{g}_K)$. Since $c_{0,0}$ is an invertible element of K, $q_1, \dots, q_\ell$ are algebraically independent elements of $\mathrm{S}(\mathfrak{g}_K)^{G_K}$ so that $q_1, \dots, q_\ell$ are homogeneous generators of $\mathrm{S}(\mathfrak{g}_K)^{G_K}$ because of their degrees. Since c_0 is an invertible element of K, ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent over K. Moreover, $(a_1, \dots, a_\ell)$ is an element of $\mathrm{S}_{d-d_1}(\mathfrak{g}^e_K) \times \cdots \times \mathrm{S}_{d-d_\ell}(\mathfrak{g}^e_K)$ such that

$$a_1({}^e q_1) + \cdots + a_\ell({}^e q_\ell) = 0.$$

Denote again by J the ideal of $\mathrm{S}(\mathfrak{g}_K)$ generated by ${}^e q_1, \dots, {}^e q_\ell$ and denote by J_i its intersection with $\mathrm{S}_i(\mathfrak{g}_K)$ for all nonnegative integer i. Then $(a_1, \dots, a_\ell)$ is not in $J_{d-d_1} \times \cdots \times J_{d-d_\ell}$ since c_1 is invertible in K. As p

is bigger than d the restriction to $\mathrm{S}_{d-d_1}(\mathfrak{g}^e_K)\times\cdots\times\mathrm{S}_{d-d_\ell}(\mathfrak{g}^e_K)$ of the quotient map $(\mathrm{S}(\mathfrak{g}^e_K))^\ell \to (\mathrm{S}_e(\mathfrak{g}^e_K))^\ell$ is injective, whence a contradiction by Corollary 4.9,(ii). As a result, for $a_1,\dots,a_\ell$ in $\mathrm{S}(\mathfrak{g}^e)$ such that

$$a_1({}^eq_1)+\cdots+a_\ell({}^eq_\ell)=0,$$

$a_1,\dots,a_\ell$ are all in J. So by Proposition 3.7, the nullvariety of ${}^eq_1,\dots,{}^eq_\ell$ in $\mathfrak{g}^f$ has codimension ℓ. Then ${}^eq_1,\dots,{}^eq_\ell$ is a regular sequence in $\mathrm{S}(\mathfrak{g}^e)$, e is a good element of $\mathfrak{g}$ and $\mathrm{S}(\mathfrak{g}^e)$ is a free extension of the polynomial algebra $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ by Proposition 3.2.

5. Consequences of Theorem 2 for the simple classical Lie algebras

This section concerns applications of Theorem 2 (or Theorem 4.1) to the simple classical Lie algebras.

5.1. The first consequence of Theorem 4.1 is the following.

Theorem 5.1. *Assume that $\mathfrak{g}$ is simple of type* **A** *or* **C**. *Then all the elements of $\mathfrak{g}$ are good.*

Proof. This follows from [PPY07, Thm. 4.2 and 4.4], Theorem 4.1 and Proposition 3.4. □

5.2. In this subsection and the next one, $\mathfrak{g}$ is assumed to be simple of type **B** or **D**. More precisely, we assume that $\mathfrak{g}$ is the simple Lie algebra $\mathfrak{so}(\mathbb{V})$ for some vector space $\mathbb{V}$ of dimension $2\ell+1$ or 2ℓ. Then $\mathfrak{g}$ is embedded into $\tilde{\mathfrak{g}}:=\mathfrak{gl}(\mathbb{V})=\mathrm{End}(\mathbb{V})$. For x an endomorphism of $\mathbb{V}$ and for $i\in\{1,\dots,\dim\mathbb{V}\}$, denote by $Q_i(x)$ the coefficient of degree $\dim\mathbb{V}-i$ of the characteristic polynomial of x. Then, for any x in $\mathfrak{g}$, $Q_i(x)=0$ whenever i is odd. Define a generating family $(q_1,\dots,q_\ell)$ of the algebra $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ as follows. For $i=1,\dots,\ell-1$, set $q_i:=Q_{2i}$. If $\dim\mathbb{V}=2\ell+1$, set $q_\ell=Q_{2\ell}$ and if $\dim\mathbb{V}=2\ell$, let q_ℓ be a homogeneous element of degree ℓ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ such that $Q_{2\ell}=q_\ell^2$.

Let (e,h,f) be an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$. Following the notations of Subsection 3.2, for $i\in\{1,\dots,\ell\}$, denote by ${}^eq_i$ the initial homogeneous component of the restriction to $\mathfrak{g}^f$ of the polynomial function $x\mapsto q_i(e+x)$, and by δ_i the degree of ${}^eq_i$. According to [PPY07, Thm. 2.1], ${}^eq_1,\dots,{}^eq_\ell$ are algebraically independent if and only if

$$\dim\mathfrak{g}^e+\ell-2(\delta_1+\cdots+\delta_\ell)=0.$$

Our first aim in this subsection is to describe the sum $\dim\mathfrak{g}^e+\ell-2(\delta_1+\cdots+\delta_\ell)$ in term of the partition of $\dim\mathbb{V}$ associated with e.

Remark 5.2. The sequence of the degrees $(\delta_1,\dots,\delta_\ell)$ is described by [PPY07, Rem. 4.2].

For $\lambda=(\lambda_1,\dots,\lambda_k)$ a sequence of positive integers, with $\lambda_1\geqslant\cdots\geqslant\lambda_k$, set:

$$|\lambda|:=k,\qquad r(\lambda):=\lambda_1+\cdots+\lambda_k.$$

Assume that the partition λ of $r(\lambda)$ is associated with a nilpotent orbit of $\mathfrak{so}(\Bbbk^{r(\lambda)})$. Then the even integers of λ have an even multiplicity, [CMc93, §5.1]. Thus k and $r(\lambda)$ have the same parity. Moreover, there is an involution $i\mapsto i'$ of $\{1,\dots,k\}$ such that $i=i'$ if λ_i is odd, and $i'\in\{i-1,i+1\}$ if λ_i is even. Set:

$$S(\lambda):=\sum_{i=i',\,i\text{ odd}} i-\sum_{i=i',\,i\text{ even}} i$$

and denote by n_λ the number of even integers in the sequence λ.

From now on, assume that λ is the partition of $\dim\mathbb{V}$ associated with the nilpotent orbit $G(e)$.

Lemma 5.3. (i) *If* $\dim \mathbb{V}$ *is odd, i.e., k is odd, then*

$$\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = \frac{n_\lambda - k - 1}{2} + S(\lambda).$$

(ii) *If* $\dim \mathbb{V}$ *is even, i.e., k is even, then*

$$\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = \frac{n_\lambda + k}{2} + S(\lambda).$$

Proof. (i) If $\dim \mathbb{V}$ is odd, then by [PPY07, §4.4, (14)],

$$2(\delta_1 + \cdots + \delta_\ell) = \dim \mathfrak{g}^e + \frac{\dim \mathbb{V}}{2} + \frac{k - n_\lambda}{2} - S(\lambda),$$

whence

$$\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = \frac{n_\lambda - k - 1}{2} + S(\lambda)$$

since $\dim \mathbb{V} = 2\ell + 1$.

(ii) If $\dim \mathbb{V}$ is even, then $\delta_\ell = k/2$ by [PPY07, Rem. 4.2] and by [PPY07, §4.4, (14)],

$$2(\delta_1 + \cdots + \delta_\ell) + k = \dim \mathfrak{g}^e + \frac{\dim \mathbb{V}}{2} + \frac{k - n_\lambda}{2} - S(\lambda)$$

whence

$$\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = \frac{n_\lambda + k}{2} + S(\lambda)$$

since $\dim \mathbb{V} = 2\ell$. □

The sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ verifies one of the following five conditions:

1) λ_k and λ_{k-1} are odd,
2) λ_k and λ_{k-1} are even,
3) $k > 3$, λ_k and λ_1 are odd and λ_i is even for any $i \in \{2, \ldots, k-1\}$,
4) $k > 4$, λ_k is odd and there is $k' \in \{2, \ldots, k-2\}$ such that $\lambda_{k'}$ is odd and λ_i is even for any $i \in \{k'+1, \ldots, k-1\}$,
5) $k = 1$ or λ_k is odd and λ_i is even for any $i < k$.

For example, $(4,4,3,1)$ verifies Condition (1); $(6,6,5,4,4)$ verifies Condition (2); $(7,6,6,4,4,4,4,3)$ verifies Condition (3); $(8,8,7,5,4,4,2,2,3)$ verifies Condition (4) with $k' = 4$; (9) and $(6,6,4,4,3)$ verify Condition (5). If $k = 2$, then one of the conditions (1) or (2) is satisfied.

Definition 5.4. Define a sequence λ^* of positive integers, with $|\lambda^*| \leqslant |\lambda|$, as follows:

- if $k = 2$ or if Condition (3) or (5) is verified, then set $\lambda^* = \lambda$,
- if Condition (1) or (2) is verified, then set:

$$\lambda^* := (\lambda_1, \ldots, \lambda_{k-2}),$$

- if $k > 3$ and if the Condition (4) is verified, then set

$$\lambda^* := (\lambda_1, \ldots, \lambda_{k'-1}).$$

In any case, the partition of $r(\lambda^*)$ corresponding to λ^* is associated with a nilpotent orbit of $\mathfrak{so}(\mathbb{k}^{r(\lambda^*)})$. Recall that n_λ is the number of even integers in the sequence λ.

Definition 5.5. Denote by d_λ the integer defined by:

- if $k = 2$, then $d_\lambda := n_\lambda$,
- if $k > 2$ and if Condition (1) or (4) is verified, then $d_\lambda := d_{\lambda^*}$,
- if $k > 2$ and if Condition (2) is verified, then $d_\lambda := d_{\lambda^*} + 2$,

- if $k > 2$ and if Condition (3) is verified, then $d_\lambda := 0$,
- if Condition (5) is verified, then $d_\lambda := 0$.

Lemma 5.6. (i) *Assume that k is odd. If Condition* (1), (2) *or* (5) *is verified, then*

$$\frac{n_\lambda - k - 1}{2} + S(\lambda) = \frac{n_{\lambda^*} - |\lambda^*| - 1}{2} + S(\lambda^*).$$

Otherwise,

$$\frac{n_\lambda - k - 1}{2} + S(\lambda) = \frac{n_{\lambda^*} - |\lambda^*| - 1}{2} + S(\lambda^*) + k - |\lambda^*| - 2.$$

(ii) *If k is even, then*

$$\frac{n_\lambda + k}{2} + S(\lambda) = \frac{n_{\lambda^*} + |\lambda^*|}{2} + S(\lambda^*) + d_\lambda - d_{\lambda^*}.$$

Proof. (i) If Condition (3) or (5) is verified, there is nothing to prove. If Condition (1) is verified,

$$n_\lambda = n_{\lambda^*}, \qquad S(\lambda) = S(\lambda^*) + 1.$$

Then

$$\frac{n_\lambda - k - 1}{2} + S(\lambda) = \frac{n_{\lambda^*} - |\lambda^*| - 1}{2} - 1 + S(\lambda^*) + 1$$

whence the assertion. If Condition (2) is verified,

$$n_\lambda = n_{\lambda^*} + 2, \qquad S(\lambda) = S(\lambda^*).$$

Then,

$$\frac{n_\lambda - k - 1}{2} + S(\lambda) = \frac{n_{\lambda^*} - |\lambda^*| - 1}{2} + S(\lambda^*)$$

whence the assertion. If Condition (4) is verified,

$$n_\lambda = n_{\lambda^*} + k - |\lambda^*| - 2, \qquad S(\lambda) = S(\lambda^*) + k - |\lambda^*| - 1.$$

Then,

$$\frac{n_\lambda - k - 1}{2} + S(\lambda) = \frac{n_{\lambda^*} - |\lambda^*| - 1}{2} - 1 + S(\lambda^*) + k - |\lambda^*| - 1$$

whence the assertion.

(ii) If $k = 2$ or if $k > 2$ and Condition (3) or (5) is verified, there is nothing to prove. Let us suppose that $k > 3$. If Condition (1) is verified,

$$n_\lambda = n_{\lambda^*}, \qquad S(\lambda) = S(\lambda^*) - 1.$$

Then

$$\frac{n_\lambda + k}{2} + S(\lambda) = \frac{n_{\lambda^*} + |\lambda^*|}{2} + 1 + S(\lambda^*) - 1$$

whence the assertion since $d_\lambda = d_{\lambda^*}$. If Condition (2) is verified,

$$n_\lambda = n_{\lambda^*} + 2, \qquad S(\lambda) = S(\lambda^*).$$

Then,

$$\frac{n_\lambda + k}{2} + S(\lambda) = \frac{n_{\lambda^*} + |\lambda^*|}{2} + 2 + S(\lambda^*)$$

whence the assertion since $d_\lambda - d_{\lambda^*} = 2$. If Condition (4) is verified,

$$n_\lambda = n_{\lambda^*} + k - |\lambda^*| - 2, \qquad S(\lambda) = S(\lambda^*) + |\lambda^*| + 1 - k.$$

Then,

$$\frac{n_\lambda + k}{2} + S(\lambda) = \frac{n_{\lambda^*} + |\lambda^*|}{2} + k - |\lambda^*| - 1 + S(\lambda^*) + |\lambda^*| - k + 1$$

whence the assertion since $d_\lambda = d_{\lambda^*}$. □

Lemma 5.7. (i) *If λ_1 is odd and if λ_i is even for $i \geqslant 2$, then* $\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = 0$.

(ii) *If k is odd, then* $\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = n_\lambda - d_\lambda$.

(iii) *If k is even, then* $\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = d_\lambda$.

Proof. (i) By the hypothesis, $n_\lambda = k - 1$ and $S(\lambda) = 1$, whence the assertion by Lemma 5.3,(i).

(ii) Let us prove the assertion by induction on k. For $k = 3$, if λ_1 and λ_2 are even, $n_\lambda = 2$, $d_\lambda = 0$ and $S(\lambda) = 3$, whence the equality by Lemma 5.3,(i). Assume that $k > 3$ and suppose that the equality holds for the integers smaller than k. If Condition (1) or (2) is verified, then by Lemma 5.3,(i), Lemma 5.6,(i) and by induction hypothesis,

$$\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = n_{\lambda^*} - d_{\lambda^*}.$$

But if Condition (1) or (2) is verified, then $n_\lambda - d_\lambda = n_{\lambda^*} - d_{\lambda^*}$. If Condition (5) is verified, then

$$n_\lambda = k - 1, \qquad S(\lambda) = k, \qquad d_\lambda = 0,$$

whence the equality by Lemma 5.3,(i). Let us suppose that Condition (4) is verified. By Lemma 5.3,(i), Lemma 5.6,(i) and by induction hypothesis,

$$\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = n_{\lambda^*} - d_{\lambda^*} + k - |\lambda^*| - 2 = n_\lambda - d_\lambda$$

whence the assertion since Condition (3) is never verified when k is odd.

(iii) The statement is clear for $k = 2$ by Lemma 5.3,(ii). Indeed, if Condition (1) is verified, then $d_\lambda = n_\lambda = 0$ and $S(\lambda) = -1$ and if Condition (2) is verified, then $d_\lambda = n_\lambda = 2$ and $S(\lambda) = 0$. If Condition (3) is verified, $n_\lambda = k - 2$ and $S(\lambda) = 1 - k$, whence the statement by Lemma 5.3,(ii). When Condition (4) is verified, by induction on $|\lambda|$, the statement results from Lemma 5.3,(ii) and Lemma 5.6,(ii), whence the assertion since Condition (5) is never verified when k is even. □

Corollary 5.8. (i) *If λ_1 is odd and if λ_i is even for all $i \geqslant 2$, then e is good.*

(ii) *If k odd and if $n_\lambda = d_\lambda$, then e is good. In particular, if $\mathfrak{g}$ is of type* **B**, *then the even nilpotent elements of $\mathfrak{g}$ are good.*

(iii) *If k even and if $d_\lambda = 0$, then e is good. In particular, if $\mathfrak{g}$ is of type* **D** *and of odd rank, then the even nilpotent elements of $\mathfrak{g}$ are good.*

Proof. As it has been already noticed, by [PPY07, Thm. 2.1], the polynomials ${}^eq_1, \ldots, {}^eq_\ell$ are algebraically independent if and only if

$$\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = 0.$$

So, by Theorem 4.1 and Lemma 5.7, if either λ_1 is odd and λ_i is even for all $i \geqslant 2$, or if k is odd and $n_\lambda = d_\lambda$, or if k is even and $d_\lambda = 0$, then e is good.

Suppose that e is even. Then the integers $\lambda_1, \ldots, \lambda_k$ have the same parity, cf. e.g. [C85, §1.3.1]. Moreover, $n_\lambda = d_\lambda = 0$ whenever $\lambda_1, \ldots, \lambda_k$ are all odd (cf. Definition 5.5). This in particular occurs if either $\mathfrak{g}$ is of type **B**, or if $\mathfrak{g}$ is of type **D** with odd rank. □

Remark 5.9. The fact that even nilpotent elements of $\mathfrak{g}$ are good if either $\mathfrak{g}$ is of type **B**, or is $\mathfrak{g}$ is of type **D** with odd rank, was already observed by O. Yakimova in [Y09, Cor. 8.2] with a different formulation.

Definition 5.10. A sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ is said to be *very good* if $n_\lambda = d_\lambda$ whenever k is odd and if $d_\lambda = 0$ whenever k is even. A nilpotent element of $\mathfrak{g}$ is said to be *very good* if it is associated with a very good partition of $\dim \mathbb{V}$.

According to Corollary 5.8, if e is very good then e is good. The following lemma characterizes the very good sequences.

Lemma 5.11. (i) *If k is odd then λ is very good if and only if λ_1 is odd and if $(\lambda_2, \dots, \lambda_k)$ is a concatenation of sequences verifying Conditions* (1) *or* (2) *with $k = 2$.*

(ii) *If k is even then λ is very good if and only if λ is a concatenation of sequences verifying Condition* (3) *or Condition* (1) *with $k = 2$.*

For example, the partitions $(5, 3, 3, 2, 2)$ and $(7, 5, 5, 4, 4, 3, 1, 1)$ of 15 and 30 respectively are very good.

Proof. (i) Assume that λ_1 is odd and that $(\lambda_2, \dots, \lambda_k)$ is a concatenation of sequences verifying Conditions (1) or (2) with $k = 2$. So, if $k > 1$, then $n_\lambda - d_\lambda = n_{\lambda^*} - d_{\lambda^*}$. Then, a quick induction shows that $n_\lambda - d_\lambda = n_{(\lambda_1)} - d_{(\lambda_1)} = 0$ since λ_1 is odd. The statement is clear for $k = 1$.

Conversely, assume that $n_\lambda - d_\lambda = 0$. If λ verifies Conditions (1) or (2), then $n_\lambda - d_\lambda = n_{\lambda^*} - d_{\lambda^*}$ and $|\lambda^*| < |\lambda|$. So, one can assume that λ does not verify Conditions (1) or (2). Since k is odd, λ cannot verify Condition (3). If λ verifies Condition (4), then $n_\lambda - d_\lambda = n_\lambda - d_{\lambda^*} > n_{\lambda^*} - d_{\lambda^*} \geqslant 0$. This is impossible since $n_\lambda - d_\lambda = 0$. If λ verifies Condition (5), then $n_\lambda - d_\lambda = n_\lambda$. So, $n_\lambda - d_\lambda = 0$ if and only if $k = 1$. Thereby, the direct implication is proven.

(ii) Assume that λ is a concatenation of sequences verifying Condition (3) or Condition (1) with $k = 2$. In particular, λ does not verify Condition (2). Moreover, Condition (5) is not verified since k is even. Then $d_\lambda = 0$ by induction on $|\lambda|$, whence e is very good.

Conversely, suppose that $d_\lambda = 0$. If $k = 2$, Condition (1) is verified and if $k = 4$, then either Condition (3) is verified, or $\lambda_1, \dots, \lambda_4$ are all odd. Suppose $k > 4$. Condition (2) is not verified since $d_\lambda = d_{\lambda_*} + 2$ in this case. If Condition (1) is verified then $d_{\lambda_*} = 0$ and λ is a concatenation of λ^* and $(\lambda_{k-1}, \lambda_k)$. If Condition (4) is verified, then $d_{\lambda_*} = 0$ and λ is a concatenation of λ_* and a sequence verifying Condition (3), whence the assertion by induction on $|\lambda|$ since Condition (5) is not verified when k is even. $\square$

5.3. Assume in this subsection that $\lambda = (\lambda_1, \dots, \lambda_k)$ verifies the following condition:

($*$) *For some $k' \in \{2, \dots, k\}$, λ_i is even for all $i \leqslant k'$, and $(\lambda_{k'+1}, \dots, \lambda_k)$ is very good.*

In particular, k' is even and λ is not very good by Lemma 5.11. For example, the sequences $\lambda = (6, 6, 4, 4, 3, 2, 2)$ and $(6, 6, 4, 4, 3, 3, 3, 2, 2, 1)$ satisfy the condition ($*$) with $k' = 4$. Define a sequence $\nu = (\nu_1, \dots, \nu_k)$ of integers of $\{1, \dots, \ell\}$ by

$$\forall\, i \in \{1, \dots, k'\}, \qquad \nu_i := \frac{\lambda_1 + \cdots + \lambda_i}{2}.$$

If $k' = k$, then $\nu_k = (\lambda_1 + \cdots + \lambda_k)/2 = r(\lambda)/2 = \ell$. Define elements $p_1, \dots, p_{k'}$ of $\mathrm{S}(\mathfrak{g}^e)$ as follows:

- if $k' < k$, set for $i \in \{1, \dots, k'\}$, $p_i := {}^e q_{\nu_i}$,
- if $k' = k$, set for $i \in \{1, \dots, k' - 1\}$, $p_i := {}^e q_{\nu_i}$ and set $p_k := ({}^e q_{\nu_k})^2$. In this case, set also $\tilde{p}_k := {}^e q_{\nu_k}$.

Remind that δ_i is the degree of ${}^e q_i$ for $i = 1, \dots, \ell$. The following lemma is a straightforward consequence of [PPY07, Rem. 4.2]:

Lemma 5.12. (i) *For all $i \in \{1, \dots, k'\}$,* $\deg\, p_i = i$.

(ii) *Set $\nu_0 := 0$. Then for $i \in \{1, \dots, k'\}$ and $r \in \{1, \dots, \nu_{k'} - 1\}$,*

$$\delta_r = i \iff \nu_{i-1} < r \leqslant \nu_i,$$

and $\delta_\ell = k/2$. In particular, for $r \in \{1, \dots, \nu_{k'} - 2\}$, $\delta_r < \delta_{r+1}$ if and only if r is a value of the sequence ν.

Example 5.13. Consider the partition $\lambda = (8, 8, 4, 4, 4, 4, 2, 2, 1, 1)$ of 38. Then $k = 10$, $k' = 8$ and $\nu = (4, 8, 10, 12, 14, 16, 17, 18)$. We represent in Table 1 the degrees of the polynomials $p_1, \dots, p_8$ and

	${}^e q_4 = p_1$	${}^e q_8 = p_2$						
	${}^e q_3$	${}^e q_7$						
	${}^e q_2$	${}^e q_6$	${}^e q_{10} = p_3$	${}^e q_{12} = p_4$	${}^e q_{14} = p_5$	${}^e q_{16} = p_6$		
	${}^e q_1$	${}^e q_5$	${}^e q_9$	${}^e q_{11}$	${}^e q_{13}$	${}^e q_{15}$	${}^e q_{17} = p_7$	${}^e q_{18} = p_8$
degrees	1	2	3	4	5	6	7	8

TABLE 1.

${}^e q_1, \dots, {}^e q_{18}$. Note that $\deg {}^e q_{19} = 5$. In the table, the common degree of the polynomials appearing on the ith column is i.

Let $\mathfrak{s}$ be the subalgebra of $\mathfrak{g}$ generated by e, h, f and decompose $\mathbb{V}$ into simple $\mathfrak{s}$-modules $\mathbb{V}_1, \dots, \mathbb{V}_k$ of dimension $\lambda_1, \dots, \lambda_k$ respectively. One can order them so that for $i \in \{1, \dots, k'/2\}$, $\mathbb{V}_{(2(i-1)+1)'} = \mathbb{V}_{2i}$. For $i \in \{1, \dots, k\}$, denote by e_i the restriction to $\mathbb{V}_i$ of e and set $\varepsilon_i := e_i^{\lambda_i - 1}$. Then e_i is a regular nilpotent element of $\mathfrak{gl}(\mathbb{V}_i)$ and $(\mathrm{ad}\, h)\varepsilon_i = 2(\lambda_i - 1)\varepsilon_i$.

For $i \in \{1, \dots, k'/2\}$, set

$$\mathbb{V}[i] := \mathbb{V}_{2(i-1)+1} + \mathbb{V}_{2i}$$

and set

$$\mathbb{V}[0] := \mathbb{V}_{k'+1} \oplus \cdots \oplus \mathbb{V}_k.$$

Then for $i \in \{0, 1, \dots, k'/2\}$, denote by $\mathfrak{g}_i$ the simple Lie algebra $\mathfrak{so}(\mathbb{V}[i])$. The elements of $\mathfrak{g}^e$ and $\mathfrak{g}^f$ stabilize $\mathbb{V}[i]$. In particular, for $i \in \{1, \dots, k'/2\}$, $e_{2(i-1)+1} + e_{2i}$ is an even nilpotent element of $\mathfrak{g}_i$ with Jordan blocks of size $(\lambda_{2(i-1)+1}, \lambda_{2i})$. Let $i \in \{1, \dots, k'/2\}$ and set,

$$z_i := \varepsilon_{2(i-1)+1} + \varepsilon_{2i}.$$

Then z_i lies in the center of $\mathfrak{g}^e$ and

$$(\mathrm{ad}\, h) z_i = 2(\lambda_{2(i-1)+1} - 1) z_i = 2(\lambda_{2i} - 1) z_i.$$

Moreover, $2(\lambda_{2i} - 1)$ is the highest weight of $\mathrm{ad}\, h$ acting on $\mathfrak{g}_i^e := \mathfrak{g}_i \cap \mathfrak{g}^e$, and the intersection of the $2(\lambda_{2i} - 1)$-eigenspace of $\mathrm{ad}\, h$ with $\mathfrak{g}_i^e$ is spanned by z_i, see for instance [Y09, §1]. Set

$$\overline{\mathfrak{g}} \quad := \quad \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{k'/2} \quad = \quad \mathfrak{so}(\mathbb{V}[0]) \oplus \mathfrak{so}(\mathbb{V}[1]) \oplus \cdots \oplus \mathfrak{so}(\mathbb{V}[k'/2])$$

and denote by $\overline{\mathfrak{g}}^e$ (resp. $\overline{\mathfrak{g}}^f$) the centralizer of e (resp. f) in $\overline{\mathfrak{g}}$. For $p \in \mathrm{S}(\mathfrak{g}^e)$, denote by $\overline{p}$ its restriction to $\overline{\mathfrak{g}}^f \simeq (\overline{\mathfrak{g}}^e)^*$; it is an element of $\mathrm{S}(\overline{\mathfrak{g}}^e)$. Our goal is to describe the elements $\overline{p}_1, \dots, \overline{p}_{k'}$ (see Proposition 5.18). The motivation comes from Lemma 5.14.

Let G^e be the centralizer of e in the adjoint group G of $\mathfrak{g}$, and G_0^e its identity component. Let $\mathfrak{g}^f_{\mathrm{reg}}$ (resp. $\overline{\mathfrak{g}}^f_{\mathrm{reg}}$) be the set of elements $x \in \mathfrak{g}^f$ (resp. $\overline{\mathfrak{g}}^f$) such that x is a regular linear form on $\mathfrak{g}^e$ (resp. $\overline{\mathfrak{g}}^e$).

Lemma 5.14. (i) *The intersection $\mathfrak{g}^f_{\mathrm{reg}} \cap \overline{\mathfrak{g}}^f$ is a dense open subset of $\overline{\mathfrak{g}}^f_{\mathrm{reg}}$.*

(ii) *The morphism*

$$\theta : \quad G_0^e \times \overline{\mathfrak{g}}^f \longrightarrow \mathfrak{g}^f, \quad (g, x) \longmapsto g.x$$

is a dominant morphism from $G_0^e \times \overline{\mathfrak{g}}^f$ to $\mathfrak{g}^f$.

Proof. (i) Since λ verifies the condition $(*)$, it verifies the condition (1) of the proof of [Y06, §4, Lem. 3] and so, $\mathfrak{g}^f_{\rm reg} \cap \overline{\mathfrak{g}}^f$ is a dense open subset of $\overline{\mathfrak{g}}^f$. Moreover, since $\mathfrak{g}^e$ and $\overline{\mathfrak{g}}^e$ have the same index by [Y06, Thm. 3], $\mathfrak{g}^f_{\rm reg} \cap \overline{\mathfrak{g}}^f$ is contained in $\overline{\mathfrak{g}}^f_{\rm reg}$.

(ii) Let $\mathfrak{m}$ be the orthogonal complement to $\overline{\mathfrak{g}}$ in $\mathfrak{g}$ with respect to the Killing form $\langle .\,,.\rangle$. Since the restriction to $\overline{\mathfrak{g}} \times \overline{\mathfrak{g}}$ of $\langle .\,,.\rangle$ is nondegenerate, $\mathfrak{g} = \overline{\mathfrak{g}} \oplus \mathfrak{m}$ and $[\overline{\mathfrak{g}}, \mathfrak{m}] \subset \mathfrak{m}$. Set $\mathfrak{m}^e := \mathfrak{m} \cap \mathfrak{g}^e$. Since the restriction to $\overline{\mathfrak{g}}^f \times \overline{\mathfrak{g}}^e$ of $\langle .\,,.\rangle$ is nondegenerate, we get the decomposition

$$\mathfrak{g}^e = \overline{\mathfrak{g}}^e \oplus \mathfrak{m}^e$$

and $\mathfrak{m}^e$ is the orthogonal complement to $\overline{\mathfrak{g}}^f$ in $\mathfrak{g}^e$. Moreover, $[\overline{\mathfrak{g}}^e, \mathfrak{m}^e] \subset \mathfrak{m}^e$.

By (i), $\mathfrak{g}^f_{\rm reg} \cap \overline{\mathfrak{g}}^f \neq \varnothing$. Let $x \in \mathfrak{g}^f_{\rm reg} \cap \overline{\mathfrak{g}}^f$. The tangent map at $(1_{\mathfrak{g}}, x)$ of θ is the linear map

$$\mathfrak{g}^e \times \overline{\mathfrak{g}}^f \longrightarrow \mathfrak{g}^f, \qquad (u, y) \longmapsto u.x + y,$$

where $u.$ denotes the coadjoint action of u on $\mathfrak{g}^f \simeq (\mathfrak{g}^e)^*$. The index of $\overline{\mathfrak{g}}^e$ is equal to the index of $\mathfrak{g}^e$ and $[\overline{\mathfrak{g}}^e, \mathfrak{m}^e] \subset \mathfrak{m}^e$. So, the stabilizer of x in $\overline{\mathfrak{g}}^e$ coincides with the stabilizer of x in $\mathfrak{g}^e$. In particular, $\dim \mathfrak{m}^e.x = \dim \mathfrak{m}^e$. As a result, θ is a submersion at $(1_{\mathfrak{g}}, x)$ since $\dim \mathfrak{g}^f = \dim \mathfrak{m}^e + \dim \overline{\mathfrak{g}}^f$. In conclusion, θ is a dominant morphism from $G_0^e \times \overline{\mathfrak{g}}^f$ to $\mathfrak{g}^f$. □

Let $\mu_1, \dots, \mu_m$ be the strictly decreasing sequence of the values of the sequence $\lambda_1, \dots, \lambda_{k'}$ and let $k_1, \dots, k_m$ be the multiplicity of $\mu_1, \dots, \mu_m$ respectively in this sequence. By our assumption, the integers $\mu_1, \dots, \mu_m, k_1, \dots, k_m$ are all even. Notice that $k_1 + \cdots + k_m = k'$. The set $\{1, \dots, k'\}$ decomposes into parts $K_1, \dots, K_m$ of cardinality $k_1, \dots, k_m$ respectively given by:

$$\forall\, s \in \{1, \dots, m\}, \qquad K_s := \{k_0 + \cdots + k_{s-1} + 1, \dots, k_0 + \cdots + k_s\}.$$

Here, the convention is that $k_0 := 0$.

Remark 5.15. For $s \in \{1, \dots, m\}$ and $i \in K_s$,

$$\nu_i := k_0(\frac{\mu_0}{2}) + \cdots + k_{s-1}(\frac{\mu_{s-1}}{2}) + j(\frac{\mu_s}{2}),$$

where $j = i - (k_0 + \cdots + k_{s-1})$ and $\mu_0 = 0$.

Decompose also the set $\{1, \dots, k'/2\}$ into parts $I_1, \dots, I_m$ of cardinality $k_1/2, \dots, k_m/2$ respectively, with

$$\forall\, s \in \{1, \dots, m\}, \qquad I_s := \{\frac{k_0 + \cdots + k_{s-1}}{2} + 1, \dots, \frac{k_0 + \cdots + k_s}{2}\}.$$

For $p \in \mathrm{S}(\mathfrak{g}^e)$ an eigenvector of $\mathrm{ad}\, h$, denote by $\mathrm{wt}(p)$ its $\mathrm{ad}\, h$-weight.

Lemma 5.16. *Let $s \in \{1, \dots, m\}$ and $i \in K_s$.*

(i) *Set $j = i - (k_0 + \cdots + k_{s-1})$. Then,*

$$\mathrm{wt}(\overline{p}_i) = 2(2\nu_i - i) = \sum_{l=1}^{s-1} 2k_l(\mu_l - 1) + 2j(\mu_s - 1).$$

Moreover, if $p \in \{{}^e q_1, \dots, {}^e q_{\ell-1}, ({}^e q_\ell)^2\}$ is of degree i, then $\mathrm{wt}(p) = \mathrm{wt}(\overline{p}) \leqslant 2(2\nu_i - i)$ and the equality holds if and only if $p = p_i$.

(ii) *The polynomial $\overline{p}_i$ is in $\Bbbk[z_l,\ l \in I_1 \cup \ldots \cup I_s]$.*

Proof. (i) This is a consequence of [PPY07, Lem. 4.3] (or [Y09, Thm. 6.1]), Lemma 5.12 and Remark 5.15.

(ii) Let $\tilde{\mathfrak{g}}^f$ be the centralizer of f in $\tilde{\mathfrak{g}} = \mathfrak{gl}(\mathbb{V})$, and let ${}^e\overline{Q}_{2\nu_i}$ be the initial homogeneous component of the restriction to

$$(\mathfrak{gl}(\mathbb{V}[0]) \oplus \mathfrak{gl}(\mathbb{V}[1]) \oplus \cdots \oplus \mathfrak{gl}(\mathbb{V}[k'/2])) \cap \tilde{\mathfrak{g}}^f$$

of the polynomial function $x \mapsto Q_{2\nu_i}(e+x)$. Since $\overline{p}_i \neq 0$, $\overline{p}_i$ is the restriction to $\overline{\mathfrak{g}}^f$ of ${}^e\overline{Q}_{2\nu_i}$ and one has

$$\mathrm{wt}({}^e\overline{Q}_{2\nu_i}) = \mathrm{wt}(\overline{p}_i) = 2(2\nu_i - i), \qquad \deg {}^e\overline{Q}_{2\nu_i} = \deg \overline{p}_i = i.$$

Then, by (i) and [PPY07, Lem. 4.3], ${}^e\overline{Q}_{2\nu_i}$ is a sum of monomials whose restriction to $\overline{\mathfrak{g}}^f$ is zero and of monomials of the form

$$(\varepsilon_{\varsigma^{(1)}1} \ldots \varepsilon_{\varsigma^{(1)}k_1}) \cdots (\varepsilon_{\varsigma^{(s-1)}1} \ldots \varepsilon_{\varsigma^{(s-1)}k_{s-1}})(\varepsilon_{\varsigma^{(s)}j_1} \ldots \varepsilon_{\varsigma^{(s)}j_i})$$

where $j_1 < \cdots < j_i$ are integers of K_s, and $\varsigma^{(1)}, \ldots, \varsigma^{(s-1)}, \varsigma^{(s)}$ are permutations of $K_1, \ldots, K_{s-1}, \{j_1, \ldots, j_i\}$ respectively. Hence, $\overline{p}_i$ is in $\Bbbk[z_l,\ l \in I_1 \cup \ldots \cup I_s]$. More precisely, for $l \in I_1 \cup \ldots, \cup I_s$, the element z_l appears in $\overline{p}_i$ with a multiplicity at most 2 since $z_l = \varepsilon_{2(l-1)+1} + \varepsilon_{2l}$. □

Let $s \in \{1, \ldots, m\}$ and $i \in K_s$. In view of Lemma 5.16,(ii), we aim to give an explicit formula for $\overline{p}_i$ in term of the elements $z_1, \ldots, z_{k'/2}$. Besides, according to Lemma 5.16,(ii), we can assume that $s = m$. As a first step, we state inductive formulae. If $k' > 2$, set

$$\overline{\mathfrak{g}}' := \mathfrak{so}(\mathbb{V}[1]) \oplus \cdots \oplus \mathfrak{so}(\mathbb{V}[k'/2 - 1]),$$

and let $\overline{p}'_1, \ldots, \overline{p}'_{k'}$ be the restrictions to $(\overline{\mathfrak{g}}')^f := \overline{\mathfrak{g}}' \cap \mathfrak{g}^f$ of $\overline{p}_1, \ldots, \overline{p}_{k'}$ respectively. Note that $\overline{p}'_{k'-1} = \overline{p}'_{k'} = 0$. Set by convention $k_0 := 0$, $p_0 := 1$, $p'_0 := 1$ and $p_{-1} := 0$. It will be also convenient to set

$$k^* := k_0 + \cdots + k_{m-1}.$$

Lemma 5.17. (i) *If $k_m = 2$, then*

$$\overline{p}_{k^*+1} = -2\,\overline{p}'_{k^*}\, z_{k'/2} \quad \textit{and} \quad \overline{p}_{k^*+2} = \overline{p}'_{k^*}\,(z_{k'/2})^2.$$

(ii) *If $k_m > 2$, then*

$$\overline{p}_{k^*+1} = \overline{p}'_{k^*+1} - 2\,\overline{p}'_{k^*}\, z_{k'/2}$$

and for $j = 2, \ldots, k_m$,

$$\overline{p}_{k^*+j} = \overline{p}'_{k^*+j} - 2\,\overline{p}'_{k^*+j-1}\, z_{k'/2} + \overline{p}'_{k^*+j-2}\,(z_{k'/2})^2.$$

Proof. For $i = 1, \ldots, k'/2$, let w_i be the element of $\mathfrak{g}_i^f := \mathfrak{g}_i \cap \mathfrak{g}^f$ such that

$$(\mathrm{ad}\, h) w_i = -2(\lambda_{2i} - 1) w_i \quad \text{and} \quad \det(e_i + w_i) = 1.$$

Remind that $p_i(y)$, for $y \in \mathfrak{g}^f$, is the initial homogeneous component of the coefficient of the term $T^{\dim \mathbb{V} - 2\nu_i}$ in the expression $\det(T - e - y)$. By Lemma 5.16,(ii), in order to describe $\overline{p}_i$, it suffices to compute $\det(T - e - s_1 w_1 - \cdots - s_{k'/2} w_{k'/2})$, with $s_1, \ldots, s_{k'/2}$ in $\Bbbk$.

1) To start with, consider the case $k' = k_m = 2$. By Lemma 5.16, $p_1 = a z_1$ and $p_2 = b z_1^2$ for some $a, b \in \Bbbk$. One has,

$$\det(T - e - s_1 w_1) = T^{2\mu_1} - 2 s_1 T^{\mu_1} + s_1^2.$$

As a result, $a = -2$ and $b = 1$. This proves (i) in this case.

2) Assume from now that $k' > 2$. Setting $e' := e_1 + \cdots + e_{k'/2-1}$, observe that

$$\begin{aligned}
(4) \qquad & \det(T - e - s_1 w_1 - \cdots - s_{k'/2} w_{k'/2}) \\
&= \det(T - e' - s_1 w_1 - \cdots - s_{k'/2-1} w_{k'/2-1}) \det(T - e_{k'/2} - s_{k'/2} w_{k'/2}) \\
&= \det(T - e' - s_1 w_1 - \cdots - s_{k'/2-1} w_{k'/2-1})\,(T^{2\mu_m} - 2 s_{k'/2} T^{\mu_m} + s_{k'/2}^2)
\end{aligned}$$

where the latter equality results from Step (1).

(i) If $k_m = 2$, then $k^* = k' - 2$ and the constant term in $\det(T - e' - s_1 w_1 - \cdots - s_{k'/2-1} w_{k'/2-1})$ is $\overline{p}'_{k^*}$. By Lemma 5.16,(i),

$$\mathrm{wt}(\overline{p}_{k^*+1}) = \mathrm{wt}(\overline{p}'_{k^*}) + \mathrm{wt}(z_{k'/2})$$

and $\overline{p}'_{k^*}$ is the only element appearing in the coefficients of $\det(T - e' - s_1 w_1 - \cdots - s_{k'/2-1} w_{k'/2-1})$ of this weight. Similarly,

$$\mathrm{wt}(\overline{p}_{k^*+2}) = \mathrm{wt}(\overline{p}'_{k^*}) + \mathrm{wt}((z_{k'/2})^2)$$

and $\overline{p}'_{k^*}$ is the only element appearing in the coefficients of $\det(T - e' - s_1 w_1 - \cdots - s_{k'/2-1} w_{k'/2-1})$ of this weight. As a consequence, the equalities follow.

(ii) Suppose $k_m > 2$. Then by Lemma 5.16,(i),

$$\mathrm{wt}(\overline{p}_{k^*+1}) = \mathrm{wt}(\overline{p}'_{k^*+1}) = \mathrm{wt}(\overline{p}'_{k^*}) + \mathrm{wt}(z_{k'/2}).$$

Moreover, $\overline{p}'_{k^*+1}$ and $\overline{p}'_{k^*}$ are the only elements appearing in the coefficients of $\det(T - e' - s_1 w_1 - \cdots - s_{k'/2-1} w_{k'/2-1})$ of this weight with degree $k^* + 1$ and k^* respectively. Similarly, by Lemma 5.16,(i), for $j \in \{2, \ldots, k_m\}$,

$$\mathrm{wt}(\overline{p}_{k^*+j}) = \mathrm{wt}(\overline{p}'_{k^*+j}) = \mathrm{wt}(\overline{p}'_{k^*+j-1}) + \mathrm{wt}(z_{k'/2}) = \mathrm{wt}(\overline{p}'_{k^*+j-2}) + \mathrm{wt}((z_{k'/2})^2).$$

Moreover, $\overline{p}'_{k^*+j}$, $\overline{p}'_{k^*+j-1}$ and $\overline{p}'_{k^*+j-2}$ are the only elements appearing in the coefficients of $\det(T - e' - s_1 w_1 - \cdots - s_{k'/2-1} w_{k'/2-1})$ of this weight with degree $k^* + j$, $k^* + j - 1$ and $k^* + j - 2$ respectively.

In both cases, this forces the inductive formula (ii) through the factorization (4). □

For a subset $I = \{i_1, \ldots, i_l\} \subseteq \{1, \ldots, k'/2\}$ of cardinality l, denote by $\sigma_{I,1}, \ldots, \sigma_{I,l}$ the elementary symmetric functions of $z_{i_1}, \ldots, z_{i_l}$:

$$\forall\, j \in \{1, \ldots, l\}, \qquad \sigma_{I,j} = \sum_{1 \leqslant a_1 < a_2 < \cdots < a_j \leqslant l} z_{i_{a_1}} z_{i_{a_2}} \ldots z_{i_{a_j}}.$$

Set also $\sigma_{I,0} := 1$ and $\sigma_{I,j} := 0$ if $j > l$ so that $\sigma_{I,j}$ is well defined for any nonnegative integer j. Set at last $\sigma_{I,j} := 1$ for any j if $I = \varnothing$. If $I = I_s$, with $s \in \{1, \ldots, m\}$, denote by $\sigma_j^{(s)}$, for $j \geqslant 0$, the elementary symmetric function $\sigma_{I_s, j}$.

Proposition 5.18. *Let $s \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, k_s\}$. Then*

$$\overline{p}_{k_0 + \cdots + k_{s-1} + j} = (-1)^j \, \overline{p}_{k_0 + \cdots + k_{s-1}} \sum_{r=0}^{j} \sigma_{j-r}^{(s)} \sigma_r^{(s)} = (-1)^j \, (\sigma_{k_0/2}^{(1)} \ldots \sigma_{k_{s-1}/2}^{(s-1)})^2 \sum_{r=0}^{j} \sigma_{j-r}^{(s)} \sigma_r^{(s)}.$$

Example 5.19. If $m = 1$, then $k' = k_1$ and

$$p_1 = -\sigma_1^{(1)} \sigma_0^{(1)} - \sigma_0^{(1)} \sigma_1^{(1)} = -2\sigma_1^{(1)} = -2(z_1 + \cdots + z_{k'/2}),$$

$$p_2 = \sigma_2^{(1)} \sigma_0^{(1)} + (\sigma_1^{(1)})^2 + \sigma_0^{(1)} \sigma_2^{(1)} = 2\sigma_2^{(1)} + (\sigma_1^{(1)})^2,$$

$$\cdots,$$

$$\overline{p}_{k'} = (\sigma_{k'/2}^{(1)})^2 = (z_1 z_2 \ldots z_{k'/2})^2.$$

Proof. By Lemma 5.16(ii), one can assume that $s = m$. Assume $m > 1$ and prove the statement by induction on $j \in \{1, \ldots, k_m\}$.

- If $k_m = 2$, the statement follows from Lemma 5.17,(i).
- Assume $k_m > 2$ and retain the notations of Lemma 5.17. In particular, set again

$$k^* := k_0 + \cdots + k_{m-1}.$$

For any $r \geqslant 0$, we set $\sigma'_r := \sigma_{I',r}$ where $I' = \{\frac{k^*}{2}+1, \dots, \frac{k'}{2}-1\} \subset I_m$. In particular, $\sigma'_0 = 1$ by convention. Observe that for any $r \geqslant 1$,

$$\sigma_r^{(m)} = \sigma'_r + \sigma'_{r-1} z_{k'/2}.$$

Setting $\sigma'_{-1} := 0$, the above equality remains true for $r = 0$.

Our induction hypothesis says that the formula holds for the polynomials $\overline{p}'_1, \dots, \overline{p}'_{k'-1}$. So, by Lemma 5.17,(ii), for $j \in \{2, \dots, k_m\}$,

$$\begin{aligned} \overline{p}_{k^*+j} &= \overline{p}'_{k^*+j} - 2\,\overline{p}'_{k^*+j-1}\, z_{k'/2} + \overline{p}'_{k^*+j-2}\,(z_{k'/2})^2 \\ &= \overline{p}_{k^*}((-1)^j \sum_{r=0}^{j} \sigma'_{j-r}\sigma'_r - 2(-1)^{j-1}\sum_{r=0}^{j-1}\sigma'_{j-r-1}\sigma'_r\, z_{k'/2} + (-1)^{j-2}\sum_{r=0}^{j-2}\sigma'_{j-r-2}\sigma'_r\, z_{k'/2}^2). \\ &= (-1)^j\,\overline{p}_{k^*}(\sum_{r=0}^{j}\sigma'_{j-r}\sigma'_r + 2\,(\sum_{r=0}^{j-1}\sigma'_{j-r-1}\sigma'_r)\, z_{k'/2} + (\sum_{r=0}^{j-2}\sigma'_{j-r-2}\sigma'_r)\, z_{k'/2}^2) \end{aligned}$$

since $\overline{p}'_{k^*} = \overline{p}_{k^*}$. On the other hand, one has

$$\begin{aligned} \sum_{r=0}^{j}\sigma_{j-r}^{(m)}\sigma_r^{(m)} &= \sum_{r=0}^{j}(\sigma'_{j-r} + \sigma'_{j-r-1} z_{k'/2})(\sigma'_r + \sigma'_{r-1} z_{k'/2}) \\ &= \sum_{r=0}^{j}\sigma'_{j-r}\sigma'_r + (\sum_{r=0}^{j}\sigma'_{j-r-1}\sigma'_r + \sum_{r=0}^{j}\sigma'_{j-r}\sigma'_{r-1})\, z_{k'/2} + (\sum_{r=0}^{j}\sigma'_{j-r-1}\sigma'_{r-1})\, z_{k'/2}^2 \\ &= \sum_{r=0}^{j}\sigma'_{j-r}\sigma'_r + 2\,(\sum_{r=0}^{j-1}\sigma'_{j-r-1}\sigma'_r)\, z_{k'/2} + (\sum_{r=0}^{j-2}\sigma'_{j-r-2}\sigma'_r)\, z_{k'/2}^2. \end{aligned}$$

Thereby, for any $j \in \{2, \dots, k_m\}$, we get

$$\overline{p}_{k^*+j} = (-1)^j\,\overline{p}_{k^*}\sum_{r=0}^{j}\sigma_{j-r}^{(m)}\sigma_r^{(m)}.$$

For $j = 1$, since $\overline{p}'_{k^*} = \overline{p}_{k^*}$, by Lemma 5.17,(ii), and our induction hypothesis,

$$\overline{p}_{k^*+1} = \overline{p}'_{k^*+1} - 2\,\overline{p}'_{k^*}\, z_{k'/2} = \overline{p}_{k^*}(-2\sigma'_1) - 2\,\overline{p}_{k^*}\, z_{k'/2} = \overline{p}_{k^*}(-2\sigma_1^{(m)}).$$

This proves the first equality of the proposition.

For the second one, it suffices to prove by induction on $s \in \{1, \dots, m\}$ that

$$\overline{p}_{k_0+\dots+k_{s-1}} = (\sigma_{k_0/2}^{(1)} \dots \sigma_{k_{s-1}/2}^{(s-1)})^2.$$

For $s = 1$, then $\overline{p}_{k_0+\dots+k_{s-1}} = \overline{p}_0 = 1$ and $\sigma_{\varnothing,0} = 1$ by convention. Assume $s > 2$ and the statement true for $1, \dots, s-1$. By the first equality with $j = k_s$, $\overline{p}_{k_0+\dots+k_s} = (-1)^{k_s}\,\overline{p}_{k_0+\dots+k_{s-1}}(\sigma_{k_s/2}^{(s)})^2$, whence the statement by induction hypothesis since k_s is even. □

Remark 5.20. As a by product of the previous formula, whenever $k' = k$, one obtains

$$\tilde{\overline{p}}_k = \sigma_{k_0/2}^{(1)} \dots \sigma_{k_m/2}^{(m)}.$$

For $s \in \{1, \dots, m\}$ and $j \in \{1, \dots, k_s\}$, set

$$\rho_{k_0+\dots+k_{s-1}+j} := \frac{\overline{p}_{k_0+\dots+k_{s-1}+j}}{\overline{p}_{k_0+\dots+k_{s-1}}}.$$

Proposition 5.18 says that $\rho_{k_0+\dots+k_{s-1}+j}$ is an element of $\mathrm{Frac}(\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}) \cap \mathrm{S}(\mathfrak{g}^e) = \mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$.

Lemma 5.21. *Let $s \in \{1,\dots,m\}$ and $j \in \{k_s/2+1,\dots,k_s\}$. There is a polynomial $R_j^{(s)}$ of degree j such that*

$$\rho_{k_0+\cdots+k_{s-1}+j} = R_j^{(s)}(\rho_{k_0+\cdots+k_{s-1}+1},\dots,\rho_{k_0+\cdots+k_{s-1}+k_s/2}).$$

In particular, for any $j \in \{k_1/2+1,\dots,k_1\}$, one has

$$\overline{p}_j = R_j^{(1)}(\overline{p}_1,\dots,\overline{p}_{k_1/2}).$$

Proof. 1) Prove by induction on $j \in \{1,\dots,k_s/2\}$ that for some polynomial $T_j^{(s)}$ of degree j,

$$\sigma_j^{(s)} = T_j^{(s)}(\rho_{k_0+\cdots+k_{s-1}+1},\dots,\rho_{k_0+\cdots+k_{s-1}+j}).$$

By Proposition 5.18, $\rho_{k_0+\cdots+k_{s-1}+1} = -(\sigma_1^{(s)}\sigma_0^{(s)} + \sigma_0^{(s)}\sigma_1^{(s)}) = -2\sigma_1^{(s)}$. Hence, the statement is true for $j=1$. Suppose $j \in \{2,\dots,k_s/2\}$ and the statement true for $\sigma_1^{(s)},\dots,\sigma_{j-1}^{(s)}$. Since $j \leqslant k_s/2$, $\sigma_j^{(s)} \neq 0$, and by Proposition 5.18,

$$\rho_{k_0+\cdots+k_{s-1}+j} = (-1)^j(\sigma_j^{(s)}\sigma_0^{(s)} + \sigma_0\sigma_j^{(s)}) + (-1)^j\sum_{r=1}^{j-1}\sigma_{j-r}^{(s)}\sigma_r^{(s)} = 2(-1)^j\sigma_j^{(s)} + (-1)^j\sum_{r=1}^{j-1}\sigma_{j-r}^{(s)}\sigma_r^{(s)}.$$

So, the statement for j follows from our induction hypothesis.

2) Let $j \in \{k_s/2+1,\dots,k_s\}$. Proposition 5.18 shows that $\rho_{k_0+\cdots+k_{s-1}+j}$ is a polynomial in $\sigma_1^{(s)},\dots,\sigma_{k_s/2}^{(s)}$. Hence, by Step 1), $\rho_{k_0+\cdots+k_{s-1}+j}$ is a polynomial in

$$\rho_{k_0+\cdots+k_{s-1}+1},\dots,\rho_{k_0+\cdots+k_{s-1}+k_s/2}.$$

Furthermore, by Proposition 5.18 and Step (1), this polynomial has degree j. $\square$

Remark 5.22. By Remark 5.20 and the above proof, if $k' = k$ then for some polynomial $\tilde{R}$ of degree $k_m/2$,

$$\frac{\overline{\tilde{p}}_k}{\sigma_{k_0/2}^{(1)}\cdots\sigma_{k_{m-1}/2}^{(m-1)}} = \sigma_{k_m/2}^{(m)} = \tilde{R}(\rho_{k_0+\cdots+k_{m-1}+1},\dots,\rho_{k_0+\cdots+k_{m-1}+k_m/2}).$$

Let $\mathfrak{g}^f_{\rm sing}$ be the set of nonregular elements of the dual $\mathfrak{g}^f$ of $\mathfrak{g}^e$.

Theorem 5.23. (i) *Assume that λ verifies the condition $(*)$ and that $\lambda_1 = \cdots = \lambda_{k'}$. Then e is good.*
(ii) *Assume that $k = 4$ and that $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ are even. Then e is good.*

For example, $(6,6,6,6,5,3)$ satisfies the hypothesis of (i) and $(6,6,4,4)$ satisfies the hypothesis of (ii).

Remark 5.24. If λ verifies the condition $(*)$ then by Lemma 5.7,

$$\dim\mathfrak{g}^e + \ell - 2(\delta_1+\cdots+\delta_\ell) = k'.$$

Indeed, if k is odd, then $n_\lambda - d_\lambda = n_{\lambda'} - d_{\lambda'}$ where $\lambda' = (\lambda_1,\dots,\lambda_{k'},\lambda_{k'+1})$ so that $n_\lambda - d_\lambda = n_{\lambda'} - d_{\lambda'} = n_{\lambda'} = k'$ since $\lambda_{k'+1}$ is odd. If k is even, then $d_\lambda = n_{\lambda'} = k'$ where $\lambda' = (\lambda_1,\dots,\lambda_{k'})$.

Proof. (i) In the previous notations, the hypothesis means that $m=1$ and $k' = k_m$. According to Lemma 5.21 and Lemma 5.14, for $j \in \{k'/2+1,\dots,k'-1\}$,

$$p_j = R_j^{(1)}(p_1,\dots,p_{k'/2}),$$

where $R_j^{(1)}$ is a polynomial of degree j. Moreover, if $k' = k$, then by Remark 5.22 and Lemma 5.14,

$$\tilde{p}_k = \tilde{R}(p_1,\dots,p_{k/2}),$$

where $\tilde{R}$ is a polynomial of degree $k/2$.

- If $k' < k$, set for any $j \in \{k'/2+1, \dots, k'\}$,

$$r_j := q_{\nu_j} - R_j^{(1)}(q_{\nu_1}, \dots, q_{\nu_{k'/2}}).$$

Then by Lemma 5.12,

$$\forall\, j \in \{k'/2+1, \dots, k'\}, \quad \deg {}^e r_j \geqslant j+1.$$

- If $k' = k$, set for $j \in \{k/2+1, \dots, k'-1\}$,

$$r_j := q_{\nu_j} - R_j^{(1)}(q_{\nu_1}, \dots, q_{\nu_{k'/2}}) \ \text{ and } \ r_k := q_{\nu_k} - \tilde{R}(q_{\nu_1}, \dots, q_{\nu_{k/2}}).$$

Then by Lemma 5.12,

$$\forall\, j \in \{k/2+1, \dots, k-1\}, \quad \deg {}^e r_j \geqslant j+1 \quad \text{ and } \quad \deg {}^e r_k \geqslant k/2+1.$$

In both cases,

$$\{q_j\ ;\ j \in \{1, \dots, \ell\} \smallsetminus \{\nu_{k'/2+1}, \dots, \nu_{k'}\}\} \ \cup\ \{r_{k'/2+1}, \dots, r_{k'}\}$$

is a homogeneous generating system of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$. Denote by $\hat{\delta}$ the sum of the degrees of the polynomials

$${}^e q_j,\ j \in \{1, \dots, \ell\} \smallsetminus \{\nu_{k'/2+1}, \dots, \nu_{k'}\}, \quad {}^e r_{k'/2+1}, \dots, {}^e r_{k'}.$$

The above discussion shows that $\hat{\delta} \geqslant \delta_1 + \cdots + \delta_\ell + k'/2$. By Remarks 5.24, one obtains

$$\dim \mathfrak{g}^e + \ell - 2\hat{\delta} \leqslant 0.$$

In conclusion, by [PPY07, Thm. 2.1] and Theorem 4.1, e is good.

(ii) In the previous notations, the hypothesis means that $k' = k = 4$. If $m = 1$ the statement is a consequence of (i). Assume that $m = 2$. Then by Proposition 5.18, $\overline{p}_1 = -2z_1$, $\overline{p}_2 = z_1^2$, $\overline{p}_3 = -2z_1^2 z_2$ and $\overline{p}_4 = (z_1 z_2)^2$. Moreover, $\overline{\tilde{p}}_4 = z_1 z_2$. Hence, by Lemma 5.14, $p_2 = \frac{1}{4}p_1^2$ and $p_3 = p_1 \tilde{p}_4$. Set $r_2 := q_{\nu_2} - \frac{1}{4}q_{\nu_1}^2$ and $r_3 := q_{\nu_3} - q_{\nu_1} q_{\nu_4}$. Then $\deg {}^e r_2 \geqslant 3$ and $\deg {}^e r_3 \geqslant 4$. Moreover,

$$\{q_1, \dots, q_\ell\} \smallsetminus \{q_{\nu_2}, q_{\nu_3}\} \cup \{r_2, r_3\}$$

is a homogeneous generating system of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$. Denoting by $\hat{\delta}$ the sum of the degrees of the polynomials

$$\{{}^e q_1, \dots, {}^e q_\ell\} \smallsetminus \{{}^e q_{\nu_2}, {}^e q_{\nu_3}\} \cup \{{}^e r_2, {}^e r_3\},$$

one obtains that $\hat{\delta} \geqslant \delta_1 + \cdots + \delta_\ell + 2$. But $\dim \mathfrak{g}^e + \ell - 2(\delta_1 + \cdots + \delta_\ell) = k' = 4$ by Remark 5.24. So, $\dim \mathfrak{g}^e + \ell - 2\hat{\delta} \leqslant 0$. In conclusion, by [PPY07, Thm. 2.1] and Theorem 4.1, e is good. $\square$

6. Examples in simple exceptional Lie algebras

We give in this section examples of good nilpotent elements in simple exceptional Lie algebras (of type $\mathbf{E}_6$, $\mathbf{F}_4$ or $\mathbf{G}_2$) which are not covered by [PPY07]. These examples are all obtained through Theorem 4.1.

Example 6.1. Suppose that $\mathfrak{g}$ has type $\mathbf{E}_6$. Let $\mathbb{V}$ be the module of highest weight the fundamental weight ϖ_1 with the notation of Bourbaki. Then $\mathbb{V}$ has dimension 27 and $\mathfrak{g}$ identifies with a subalgebra of $\mathfrak{sl}_{27}(\Bbbk)$. For x in $\mathfrak{sl}_{27}(\Bbbk)$ and for $i = 2, \dots, 27$, let $p_i(x)$ be the coefficient of T^{27-i} in $\det(T - x)$ and denote by q_i the restriction of p_i to $\mathfrak{g}$. Then $(q_2, q_5, q_6, q_8, q_9, q_{12})$ is a generating family of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ since these polynomials are algebraically independent, [Me88]. Let (e, h, f) be an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$. Then (e, h, f) is an $\mathfrak{sl}_2$-triple of $\mathfrak{sl}_{27}(\Bbbk)$. We denote by ${}^e p_i$ the initial homogeneous component of the restriction to $e + \tilde{\mathfrak{g}}^f$ of p_i where $\tilde{\mathfrak{g}}^f$ is the centralizer of f in $\mathfrak{sl}_{27}(\Bbbk)$. As usual, ${}^e q_i$ denotes the initial homogeneous component of the restriction to $e + \mathfrak{g}^f$ of q_i. For $i = 2, 5, 6, 8, 9, 12$,

$$\deg {}^e p_i \leqslant \deg {}^e q_i.$$

In some cases, from the knowledge of the maximal eigenvalue of the restriction of $\operatorname{ad} h$ to $\mathfrak{g}$ and the $\operatorname{ad} h$-weight of ${}^e p_i$, it is possible to deduce that $\deg {}^e p_i < \deg {}^e q_i$. On the other hand,

$$\deg {}^e q_2 + \deg {}^e q_5 + \deg {}^e q_6 + \deg {}^e q_8 + \deg {}^e q_9 + \deg {}^e q_{12} \leqslant \frac{1}{2}(\dim \mathfrak{g}^e + 6),$$

with equality if and only if ${}^e q_2, {}^e q_5, {}^e q_6, {}^e q_8, {}^e q_9, {}^e q_{12}$ are algebraically independent. From this, it is possible to deduce in some cases that e is good. These cases are listed in Table 2 where the nine columns are indexed in the following way:

1: the label of the orbit $G(e)$ in the Bala-Carter classification,
2: the weighted Dynkin diagram of $G(e)$,
3: the dimension of $\mathfrak{g}^e$,
4: the partition of 27 corresponding to the nilpotent element e of $\mathfrak{sl}_{27}(\Bbbk)$,
5: the degrees of ${}^e p_2, {}^e p_5, {}^e p_6, {}^e p_8, {}^e p_9, {}^e p_{12}$,
6: their $\operatorname{ad} h$-weights,
7: the maximal eigenvalue ν of the restriction of $\operatorname{ad} h$ to $\mathfrak{g}$,
8: the sum Σ of the degrees of ${}^e p_2, {}^e p_5, {}^e p_6, {}^e p_8, {}^e p_9, {}^e p_{12}$,
9: the sum $\Sigma' = \frac{1}{2}(\dim \mathfrak{g}^e + \ell)$.

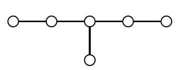
	Label		$\dim \mathfrak{g}^e$	partition	$\deg {}^e p_i$	weights	ν	Σ	Σ'
1.	E_6	2 2 2 2 2 2	6	(17,9,1)	1,1,1,1,1,1	2,8,10,14,16,22	16	6	6
2.	$E_6(a_1)$	2 2 0 2 2 2	8	(13,9,5)	1,1,1,1,1,1	2,8,10,14,16,22	16	6	7
3.	D_5	2 0 2 0 2 2	10	(11,9,5,1,1)	1,1,1,1,1,1	2,8,10,14,16,22	14	6	8
4.	$A_5 + A_1$	2 0 2 0 2 0	12	$(9,7,5^2,1)$	1,1,1,1,1,2	2,8,10,14,16,20	10	7	9
5.	$D_5(a_1)$	1 1 0 1 1 2	14	(8,7,6,3,2,1)	1,1,1,1,2,2	2,8,10,14,14,20	10	8	10
6.	A_5	2 1 0 1 2 1	14	$(9,6^2,5,1)$	1,1,1,1,1,2	2,8,10,14,16,20	10	7	10
7.	$A_4 + A_1$	1 1 0 1 1 1	16	$(7,6,5,4,3,2)$	1,1,1,2,2,2	2,8,10,12,14,20	8	9	11
8.	D_4	0 0 2 0 0 2	18	$(7^3,1^6)$	1,1,1,2,2,2	2,8,10,12,14,20	10	9	12
9.	$A_3 + 2A_1$	0 0 2 0 0 0	20	$(5^3,3^3,1^3)$	1,1,2,2,2,3	2,8,8,12,14,18	6	11	13
10.	$A_1 + 2A_2$	1 0 1 0 1 0	24	$(5,4^2,3^3,2^2,1)$	1,1,2,2,2,3	2,8,8,12,14,18	5	11	15

TABLE 2. Data for $\mathbf{E}_6$

For the orbit **1**, $\Sigma = \Sigma'$. Hence, ${}^e q_2$, ${}^e q_5$, ${}^e q_6$, ${}^e q_8$, ${}^e q_9$, ${}^e q_{12}$ are algebraically independent and by Theorem 4.1, e is good. For the orbits **2**,**3**,…,**10**, we observe that $\Sigma < \Sigma'$, i.e.,

$$\deg {}^e p_2 + \deg {}^e p_5 + \deg {}^e p_6 + \deg {}^e p_8 + \deg {}^e p_9 + \deg {}^e p_{12} < \frac{1}{2}(\dim \mathfrak{g}^e + 6).$$

So, we need some more arguments that we give below.

2. Since $16 < 22$, $\deg {}^e p_{12} < \deg {}^e q_{12}$.
3. Since $14 < 16$, $\deg {}^e p_i < \deg {}^e q_i$ for $i = 9, 12$.
4. Since $10 < 14$, $\deg {}^e p_i < \deg {}^e q_i$ for $i = 8, 9$.
5. Since $10 < 14$, $\deg {}^e p_8 < \deg {}^e q_8$. Moreover, the multiplicity of the weight 10 equals 1. So, either $\deg {}^e q_6 > 1$, or $\deg {}^e q_{12} > 2$, or ${}^e q_{12} \in \Bbbk\, {}^e q_6^2$.
6. Since $10 < 14$, $\deg {}^e p_i < \deg {}^e q_i$ for $i = 8, 9$. Moreover, the multiplicity of the weight 10 equals 1. So, either $\deg {}^e q_6 > 1$, or $\deg {}^e q_{12} > 2$, or ${}^e q_{12} \in \Bbbk\, {}^e q_6^2$.
7. Since $8 < 10$ and $2\times 8 < 20$, $\deg {}^e p_i < \deg {}^e q_i$ for $i = 6, 12$.
8. Since the center of $\mathfrak{g}^e$ has dimension 2 and the weights of h in the center are 2 and 10, $\deg {}^e p_5 < \deg {}^e q_5$. Moreover, since the weights of h in $\mathfrak{g}^e$ are $0, 2, 6, 10$, $\deg {}^e p_9 < \deg {}^e q_9$ and since the multiplicity of the weight 10 equals 1, either $\deg {}^e q_6 > 1$, or $\deg {}^e q_{12} > 2$, or ${}^e q_{12} \in \Bbbk\, {}^e q_6^2$.
9. Since $6 < 8$ and $2\times 6 < 14$, $\deg {}^e p_i < \deg {}^e q_i$ for $i = 5, 9$.
10. Since $5 < 8$, $2\times 5 < 12$ and $3\times 5 < 18$, $\deg {}^e p_i < \deg {}^e q_i$ for $i = 5, 8, 9, 12$.

In cases **2**, **3**, **4**, **7**, **9**, **10**, the discussion shows that

$$\deg {}^e q_2 + \deg {}^e q_5 + \deg {}^e q_6 + \deg {}^e q_8 + \deg {}^e q_9 + \deg {}^e q_{12} = \frac{1}{2}(\dim \mathfrak{g}^e + 6).$$

Hence, ${}^e q_2$, ${}^e q_5$, ${}^e q_6$, ${}^e q_8$, ${}^e q_9$, ${}^e q_{12}$ are algebraically independent and by Theorem 4.1, e is good. In cases **5**, **6**, **8**, if the above equality does not hold, then for some a in $\Bbbk^*$,

$$\deg {}^e q_2 + \deg {}^e q_5 + \deg {}^e q_6 + \deg {}^e q_8 + \deg {}^e q_9 + \deg {}^e (q_{12} - a q_6^2) = \frac{1}{2}(\dim \mathfrak{g}^e + 6).$$

Hence ${}^e q_2$, ${}^e q_5$, ${}^e q_6$, ${}^e q_8$, ${}^e q_9$, ${}^e (q_{12} - a q_6^2)$ are algebraically independent and by Theorem 4.1, e is good.

In addition, according to [PPY07, Thm. 0.4] and Theorem 4.1, the elements of the minimal orbit of $\mathbf{E}_6$, labelled A_1, are good. In conclusion, it remains nine unsolved nilpotent orbits in type $\mathbf{E}_6$.

Example 6.2. Suppose that $\mathfrak{g}$ is simple of type $\mathbf{F}_4$. Let $\mathbb{V}$ be the module of highest weight the fundamental weight ϖ_4 with the notation of Bourbaki. Then $\mathbb{V}$ has dimension 26 and $\mathfrak{g}$ identifies with a subalgebra of $\mathfrak{sl}_{26}(\Bbbk)$. For x in $\mathfrak{sl}_{26}(\Bbbk)$ and for $i = 2, \ldots, 26$, let $p_i(x)$ be the coefficient of T^{26-i} in $\det(T - x)$ and denote by q_i the restriction of p_i to $\mathfrak{g}$. Then (q_2, q_6, q_8, q_{12}) is a generating family of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ since these polynomials are algebraically independent, [Me88]. Let (e, h, f) be an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$. Then (e, h, f) is an $\mathfrak{sl}_2$-triple of $\mathfrak{sl}_{26}(\Bbbk)$. As in Example 6.1, in some cases, it is possible to deduce that e is good. These cases are listed in Table 3, indexed as in Example 6.1.

	Label					$\dim \mathfrak{g}^e$	partition	$\deg {}^e p_i$	weights	ν	Σ	Σ'
1.	F_4	2	2	2	2	4	(17,9)	1,1,1,1	2,10,14,22	22	4	4
2.	B_4	2	2	0	2	6	(11,9,5,1)	1,1,1,1	2,10,14,22	14	4	5
3.	$C_3 + A_1$	0	2	0	2	8	$(9, 7, 5^2)$	1,1,1,2	2,10,14,20	10	5	6
4.	C_3	1	0	1	2	10	$(9, 6^2, 5)$	1,1,1,2	2,10,14,20	10	5	7
5.	B_3	2	2	0	0	10	$(7^3, 1^5)$	1,1,2,2	2,10,12,20	10	6	7
6.	$\tilde{A}_2 + A_2$	0	2	0	0	12	$(5^3, 3^3, 1^2)$	1,2,2,3	2,8,12,18	6	8	8
7.	$B_2 + A_1$	1	0	1	0	14	$(5^2, 4^2, 3, 2^2, 1)$	1,2,2,3	2,8,12,18	6	8	9
8.	$\tilde{A}_2 + A_1$	0	1	0	1	16	$(5, 4^2, 3^3, 2^2)$	1,2,2,3	2,8,12,18	5	8	10

TABLE 3. Data for $\mathbf{F}_4$

For the orbits **2**, **3**, **4**, **5**, **7**,**8**, we observe that $\Sigma < \Sigma'$. So, we need some more arguments to conclude as in Example 6.1.

2. Since $14 < 22$, $\deg {}^e p_{12} < \deg {}^e q_{12}$.
3. Since $10 < 14$, $\deg {}^e p_8 < \deg {}^e q_8$.
4. Since $10 < 14$, $\deg {}^e p_8 < \deg {}^e q_8$. Moreover, the multiplicity of the weight 10 equals 1 so that $\deg {}^e q_6 > 1$ or $\deg {}^e q_{12} > 2$ or ${}^e q_{12} \in \Bbbk\, {}^e q_6^2$.
5. The multiplicity of the weight 10 equals 1. So, either $\deg {}^e q_6 > 1$, or $\deg {}^e q_{12} > 2$, or ${}^e q_{12} \in \Bbbk\, {}^e q_6^2$.
7. Suppose that ${}^e q_2, {}^e q_6, {}^e q_8, {}^e q_{12}$ have degree $1, 2, 2, 3$. We expect a contradiction. Since the center has dimension 2 and since the multiplicity of the weight 6 equals 1, for z of weight 6 in the center, ${}^e q_6 \in \Bbbk e z$, ${}^e q_8 \in \Bbbk z^2$, ${}^e q_{12} \in \Bbbk z^3$. So, for some a and b in $\Bbbk^*$,

$$ {}^e q_2^2\, {}^e q_8 - a\, {}^e q_6^2 = 0, \qquad {}^e q_{12}^2 - b\, {}^e q_8^3 = 0 $$

Hence, $q_2, q_6, q_2^2 q_8 - a q_6^2, q_{12}^2 - b q_8^3$ are algebraically independent element of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ such that

$$ \deg {}^e q_2 + \deg {}^e q_6 + \deg {}^e (q_2^2 q_8 - a q_6^2) + \deg {}^e (q_{12}^2 - b q_8^3) \geqslant 1 + 2 + 5 + 7 > 2 + 3 + 9 $$

whence a contradiction by [PPY07, Thm. 2.1].
8. Since $2{\times}5 < 12$ and $3{\times}5 < 18$, $\deg {}^e q_8 > \deg {}^e p_8$ and $\deg {}^e q_{12} > \deg {}^e p_{12}$.

In addition, according to [PPY07, Thm. 0.4] and Theorem 4.1, the elements of the minimal orbit of $\mathbf{F}_4$, labelled A_1, are good. In conclusion, it remains six unsolved nilpotent orbits in type $\mathbf{F}_4$.

Example 6.3. Suppose that $\mathfrak{g}$ is simple of type $\mathbf{G}_2$. Let $\mathbb{V}$ be the module of highest weight the fundamental weight ϖ_1 with the notation of Bourbaki. Then $\mathbb{V}$ has dimension 7 and $\mathfrak{g}$ identifies with a subalgebra of $\mathfrak{sl}_7(\Bbbk)$. For x in $\mathfrak{sl}_7(\Bbbk)$ and for $i = 2, \ldots, 7$, let $p_i(x)$ be the coefficient of T^{7-i} in $\det(T - x)$ and denote by q_i the restriction of p_i to $\mathfrak{g}$. Then q_2, q_6 is a generating family of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ since these polynomials are algebraically independent, [Me88]. Let (e,h,f) be an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$. Then (e,h,f) is an $\mathfrak{sl}_2$-triple of $\mathfrak{sl}_7(\Bbbk)$. In all cases, we deduce that e is good from Table 4, indexed as in Example 6.1.

	Label			$\dim \mathfrak{g}^e$	partition	$\deg {}^e p_i$	weights	ν	Σ	Σ
1.	G_2	2	2	2	(7)	1,1	2,10	10	2	2
2.	$A_1+\tilde{A}_1$	0	2	4	$(3^2,1)$	1,2	2,8	4	3	3
3.	$\tilde{A}_1$	1	0	6	$(3,2^2)$	1,3	2,6	3	4	4
4.	A_1	0	1	8	$(2^2,1^3)$	1,4	2,4	2	5	5

TABLE 4. Data for $\mathbf{G}_2$

7. OTHER EXAMPLES, REMARKS AND A CONJECTURE

This section provides examples of nilpotent elements which verify the polynomiality condition but that are not good. We also obtain an example of nilpotent element in type $\mathbf{D}_7$ which does not verify the polynomiality condition (cf. Example 7.8). Then we conclude with some remarks and a conjecture.

7.1. **Some general results.** In this subsection, $\mathfrak{g}$ is a simple Lie algebra over $\Bbbk$ and (e,h,f) is an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$. For p in $\mathrm{S}(\mathfrak{g})$, ${}^e p$ is the initial homogeneous component of the restriction of p to the Slodowy slice $e+\mathfrak{g}^f$. Recall that $\Bbbk[e+\mathfrak{g}^f]$ identifies with $\mathrm{S}(\mathfrak{g}^e)$ by the Killing form $\langle .\,,.\rangle$ of $\mathfrak{g}$.

Let η_0 be in $\mathfrak{g}^e\otimes_\Bbbk \bigwedge^2 \mathfrak{g}^f$ the bivector defining the Poisson bracket on $\mathrm{S}(\mathfrak{g}^e)$. According to the main theorem of [Pr02], $\mathrm{S}(\mathfrak{g}^e)$ is the graded algebra associated to the Kazhdan filtration of the W-algebra $\tilde{H}_e$ so that $\mathrm{S}(\mathfrak{g}^e)$ inherits a Poisson structure. Let η be in $\mathrm{S}(\mathfrak{g}^e)\otimes_\Bbbk \bigwedge^2 \mathfrak{g}^f$ the bivector defining this other Poisson structure. According to [Pr02, Prop. 6.3], η_0 is the initial homogeneous component of η. Denote by r the dimension of $\mathfrak{g}^e$ and set:

$$\omega := \eta^{(r-\ell)/2} \in \mathrm{S}(\mathfrak{g}^e)\otimes_\Bbbk \textstyle\bigwedge^{r-\ell}\mathfrak{g}^f, \qquad \omega_0 := \eta_0^{(r-\ell)/2} \in \mathrm{S}(\mathfrak{g}^e)\otimes_\Bbbk \textstyle\bigwedge^{r-\ell}\mathfrak{g}^f.$$

Then ω_0 is the initial homogeneous component of ω.

Let $v_1,\dots,v_r$ be a basis of $\mathfrak{g}^f$. For μ in $\mathrm{S}(\mathfrak{g}^e)\otimes_\Bbbk \bigwedge^i \mathfrak{g}^e$, denote by $j(\mu)$ the image of $v_1\wedge\cdots\wedge v_r$ by the right interior product of μ so that

$$j(\mu)\in \mathrm{S}(\mathfrak{g}^e)\otimes_\Bbbk \bigwedge^{r-i}\mathfrak{g}^f.$$

Lemma 7.1. *Let $q_1,\dots,q_\ell$ be some homogeneous generators of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ and let $r_1,\dots,r_\ell$ be algebraically independent homogeneous elements of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$.*

(i) *For some homogeneous element p of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$,*

$$\mathrm{d}r_1\wedge\cdots\wedge \mathrm{d}r_\ell = p\,\mathrm{d}q_1\wedge\cdots\wedge\mathrm{d}q_\ell.$$

(ii) *The following inequality holds:*

$$\sum_{i=1}^{\ell}\deg {}^e r_i \leqslant \deg {}^e p + \frac{1}{2}(\dim \mathfrak{g}^e+\ell).$$

(iii) *The polynomials* ${}^e r_1, \dots, {}^e r_\ell$ *are algebraically independent if and only if*

$$\sum_{i=1}^{\ell} \deg {}^e r_i = \deg {}^e p + \frac{1}{2}(\dim \mathfrak{g}^e + \ell).$$

Proof. (i) Since $q_1, \dots, q_\ell$ are generators of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, for $i \in \{1, \dots, \ell\}$, $r_i = R_i(q_1, \dots, q_\ell)$ where R_i is a polynomial in ℓ indeterminates, whence the assertion with

$$p = \det(\frac{\partial R_i}{\partial q_j},\ 1 \leqslant i, j \leqslant \ell).$$

(ii) Remind that for p in $\mathrm{S}(\mathfrak{g})$, $\kappa(p)$ denotes the restriction to $\mathfrak{g}^f$ of the polynomial function $x \mapsto p(e + x)$. According to [PPY07, Thm. 1.2],

$$j(\mathrm{d}\kappa(q_1) \wedge \cdots \wedge \mathrm{d}\kappa(q_\ell)) = a\omega$$

for some a in $\mathbb{k}^*$. Hence by (i),

$$j(\mathrm{d}\kappa(r_1) \wedge \cdots \wedge \mathrm{d}\kappa(r_\ell)) = a\kappa(p)\omega.$$

The initial homogeneous component of the right-hand side is $a\,{}^e p\omega_0$ and the degree of the initial homogeneous component of the left-hand side is at least

$$\deg {}^e r_1 + \cdots + \deg {}^e r_\ell - \ell.$$

The assertion follows since ω_0 has degree

$$\frac{1}{2}(\dim \mathfrak{g}^e - \ell).$$

(iii) If ${}^e r_1, \dots, {}^e r_\ell$ are algebraically independent, then the degree of the initial homogeneous component of $j(\mathrm{d}r_1 \wedge \cdots \wedge \mathrm{d}r_\ell)$ equals

$$\deg {}^e r_1 + \cdots + \deg {}^e r_\ell - \ell$$

whence

$$\deg {}^e r_1 + \cdots + \deg {}^e r_\ell = \deg {}^e p + \frac{1}{2}(\dim \mathfrak{g}^e + \ell)$$

by the proof of (ii). Conversely, if the equality holds, then

$$j(\mathrm{d}\,{}^e r_1 \wedge \cdots \wedge \mathrm{d}\,{}^e r_\ell) = a\,{}^e p\omega_0 \tag{5}$$

by the proof of (ii). In particular, ${}^e r_1, \dots, {}^e r_\ell$ are algebraically independent. □

Corollary 7.2. *For* $i = 1, \dots, \ell$, *let* $r_i := R_i(q_1, \dots, q_i)$ *be a homogeneous element of* $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ *such that* $\dfrac{\partial R_i}{\partial q_i} \neq 0$. *Then* ${}^e r_1, \dots, {}^e r_\ell$ *are algebraically independent if and only if*

$$\deg {}^e r_1 + \cdots + \deg {}^e r_\ell = \sum_{i=1}^{\ell} \deg {}^e p_i + \frac{1}{2}(\dim \mathfrak{g}^e + \ell)$$

with $p_i = \dfrac{\partial R_i}{\partial q_i}$ *for* $i = 1, \dots, \ell$.

Proof. Since $\dfrac{\partial R_i}{\partial q_i} \neq 0$ for all i, $r_1, \dots, r_\ell$ are algebraically independent and

$$\mathrm{d}r_1 \wedge \cdots \wedge \mathrm{d}r_\ell = \prod_{i=1}^{\ell} \frac{\partial R_i}{\partial q_i} \mathrm{d}q_1 \wedge \cdots \wedge \mathrm{d}q_\ell$$

whence the corollary by Lemma 7.1,(iii). □

Remind that $\mathfrak{g}^f_{\rm sing}$ is the set of nonregular elements of the dual $\mathfrak{g}^f$ of $\mathfrak{g}^e$. If $\mathfrak{g}^f_{\rm sing}$ has codimension at least 2 in $\mathfrak{g}^f$, we will say that $\mathfrak{g}^e$ is *nonsingular*.

Corollary 7.3. *Let $q_1,\dots,q_\ell$, $r_1,\dots,r_\ell$, p be as in Lemma 7.1 and such that ${}^er_1,\dots,{}^er_\ell$ are algebraically independent.*

(i) *If ${}^ep$ is a greatest common divisor of ${\rm d}\,{}^er_1\wedge\cdots\wedge{\rm d}\,{}^er_\ell$ in ${\rm S}(\mathfrak{g}^e)\otimes_{\Bbbk}\bigwedge^\ell\mathfrak{g}^e$, then $\mathfrak{g}^e$ is nonsingular.*

(ii) *Assume that there are homogeneous polynomials $p_1,\dots,p_\ell$ in ${\rm S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ verifying the following conditions:*

1) *${}^er_1,\dots,{}^er_\ell$ are in $\Bbbk[p_1,\dots,p_\ell]$,*
2) *if d is the degree of a greatest common divisor of ${\rm d}p_1\wedge\cdots\wedge{\rm d}p_\ell$ in ${\rm S}(\mathfrak{g}^e)$, then*

$$\deg p_1+\cdots+\deg p_\ell = d+\frac{1}{2}(\dim\mathfrak{g}^e+\ell).$$

Then $\mathfrak{g}^e$ is nonsingular.

Proof. (i) Suppose that ${}^ep$ is a greatest common divisor of ${\rm d}\,{}^er_1\wedge\cdots\wedge{\rm d}\,{}^er_\ell$ in ${\rm S}(\mathfrak{g}^e)\otimes_{\Bbbk}\bigwedge^\ell\mathfrak{g}^e$. Then for some ω_1 in ${\rm S}(\mathfrak{g}^e)\otimes_{\Bbbk}\bigwedge^\ell\mathfrak{g}^e$ whose nullvariety in $\mathfrak{g}^f$ has codimension at least 2,

$${\rm d}\,{}^er_1\wedge\cdots\wedge{\rm d}\,{}^er_\ell = {}^ep\,\omega_1.$$

Therefore $j(\omega_1)=a\omega_0$ by the equality (5). Since x is in $\mathfrak{g}^f_{\rm sing}$ if and only if $\omega_0(x)=0$, we get (i).

(ii) By Condition (1),

$${\rm d}\,{}^er_1\wedge\cdots\wedge{\rm d}\,{}^er_\ell = q\,{\rm d}p_1\wedge\cdots\wedge{\rm d}p_\ell$$

for some q in ${\rm S}(\mathfrak{g})^{\mathfrak{g}}$, and for some greatest common divisor q' of ${\rm d}p_1\wedge\cdots\wedge{\rm d}p_\ell$ in ${\rm S}(\mathfrak{g}^e)\otimes_{\Bbbk}\bigwedge^\ell\mathfrak{g}^e$,

$${\rm d}p_1\wedge\cdots\wedge{\rm d}p_\ell = q'\omega_1.$$

So, by the equality (5),

$$qq'\,j(\omega_1) = a\,{}^ep\,\omega_0, \tag{6}$$

so that ${}^ep$ divides qq' in ${\rm S}(\mathfrak{g}^e)$. By Condition (2) and the equality (6), ω_0 and ω_1 have the same degree. Then qq' is in $\Bbbk^*\,{}^ep$, and for some a' in $\Bbbk^*$,

$$j(\omega_1)=a'\omega_0,$$

whence (ii), again since x is in $\mathfrak{g}^f_{\rm sing}$ if and only if $\omega_0(x)=0$. □

The following proposition is a particular case of [JS10, 5.7].

Proposition 7.4. *Suppose that $\mathfrak{g}^e$ is nonsingular.*

(i) *If there exist algebraically independent homogeneous polynomials $p_1,\dots,p_\ell$ in ${\rm S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ such that*

$$\deg p_1+\cdots+\deg p_\ell = \frac{1}{2}(\dim\mathfrak{g}^e+\ell)$$

then ${\rm S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra generated by $p_1,\dots,p_\ell$.

(ii) *Suppose that the semiinvariant elements of ${\rm S}(\mathfrak{g}^e)$ are invariant. If ${\rm S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra then it is generated by homogeneous polynomials $p_1,\dots,p_\ell$ such that*

$$\deg p_1+\cdots+\deg p_\ell = \frac{1}{2}(\dim\mathfrak{g}^e+\ell).$$

7.2. **New examples.** To produce new examples, our general strategy is to apply Proposition 7.4,(i). To that end, we first apply Corollary 7.3 in order to show that $\mathfrak{g}^e$ is nonsingular. Then, we search for independent homogeneous polynomials $p_1, \dots, p_\ell$ in $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ satisfying the condition (ii) of Corollary 7.3 with $d = 0$.

Example 7.5. Let e be a nilpotent element of $\mathfrak{so}(\mathbb{k}^{10})$ associated with the partition $(3,3,2,2)$. Then $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra but e is not good.

In this case, $\ell = 5$ and let $q_1, \dots, q_5$ be as in Subsection 5.2. The degrees of ${}^e q_1, \dots, {}^e q_5$ are $1,2,2,3,2$ respectively. By a computation performed by Maple, ${}^e q_1, \dots, {}^e q_5$ verify the algebraic relation:

$${}^e q_4^2 - 4\,{}^e q_3\,{}^e q_5^2.$$

Set:

$$r_i := \begin{cases} q_i & \text{if} \quad i = 1,2,3,5 \\ q_4^2 - 4q_3q_5^2 & \text{if} \qquad i = 4 \end{cases}$$

The polynomials $r_1, \dots, r_5$ are algebraically independent over $\mathbb{k}$ and

$$\mathrm{d}r_1 \wedge \cdots \wedge \mathrm{d}r_5 = 2\,q_4\,\mathrm{d}q_1 \wedge \cdots \wedge \mathrm{d}q_5$$

Moreover, ${}^e r_4$ has degree at least 7. Then, by Corollary 7.2, ${}^e r_1, \dots, {}^e r_5$ are algebraically independent since

$$\frac{1}{2}(\dim \mathfrak{g}^e + 5) + 3 = 14 = 1 + 2 + 2 + 2 + 7,$$

and by Lemma 7.1,(ii) and (iii), ${}^e r_4$ has degree 7.

A precise computation performed by Maple shows that ${}^e r_3 = p_3^2$ for some p_3 in the center of $\mathfrak{g}^e$, and that ${}^e r_4 = p_4\,{}^e r_5$ for some polynomial p_4 of degree 5 in $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$. Setting $p_i := {}^e r_i$ for $i = 1,2,5$, the polynomials $p_1, \dots, p_5$ are algebraically independent homogeneous polynomials of degree $1,2,1,5,2$ respectively. Furthermore, a computation performed by Maple proves that the greatest common divisors of $\mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_5$ in $S(\mathfrak{g}^e)$ have degree 0, and that p_4 is in the ideal of $S(\mathfrak{g}^e)$ generated by p_3 and p_5. So, by Corollary 7.3,(ii), $\mathfrak{g}^e$ is nonsingular, and by Proposition 7.4,(i), $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra generated by $p_1, \dots, p_5$. Moreover, e is not good since the nullvariety of $p_1, \dots, p_5$ in $(\mathfrak{g}^e)^*$ has codimension at most 4.

Example 7.6. In the same way, for the nilpotent element e of $\mathfrak{so}(\mathbb{k}^{11})$ associated with the partition $(3,3,2,2,1)$, one can show that $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra generated by polynomials of degree $1,1,2,2,7$, $\mathfrak{g}^e$ is nonsingular but e is not good.

We also obtain that for the nilpotent element e of $\mathfrak{so}(\mathbb{k}^{12})$ (resp. $\mathfrak{so}(\mathbb{k}^{13})$) associated with the partition (5,3,2,2) or (3,3,2,2,1,1) (resp. (5,3,2,2,1), (4,4,2,2,1), or (3,3,2,2,1,1,1)), $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra, $\mathfrak{g}^e$ is nonsingular but e is not good.

We can summarize our conclusions for the small ranks. Assume that $\mathfrak{g} = \mathfrak{so}(\mathbb{V})$ for some vector space $\mathbb{V}$ of dimension $2\ell + 1$ or 2ℓ and let $e \in \mathfrak{g}$ be a nilpotent element of $\mathfrak{g}$ associated with the partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of $\dim \mathbb{V}$. If $\ell \leqslant 6$, our previous results (Corollary 5.8, Lemma 5.11, Theorem 5.23, Examples 7.5 and 7.6) show that either e is good, or e is not good but $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is nevertheless a polynomial algebra and $\mathfrak{g}^e$ is nonsingular. We describe in Table 5 the partitions λ corresponding to good e, and those corresponding to the case where e is not good. The third column of the table gives the degrees of the generators in the latter case.

Remark 7.7. The above discussion shows that there are good nilpotent elements for which the codimension of $(\mathfrak{g}^e)^*_{\mathrm{sing}}$ in $(\mathfrak{g}^e)^*$ is 1. Indeed, by [PPY07, §3.9], for some nilpotent element e' in $\mathbf{B}_3$, the codimension of $(\mathfrak{g}^{e'})^*_{\mathrm{sing}}$ in $(\mathfrak{g}^{e'})^*$ is 1 but, in $\mathbf{B}_3$, all nilpotent elements are good (cf. Table 5).

Type	e is good	$S(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is polynomial, $\mathfrak{g}^e$ is nonsingular but e is not good	degrees of the generators
$\mathbf{B}_n, \mathbf{D}_n, n \leqslant 4$	any λ	$\varnothing$	
$\mathbf{B}_5$	$\lambda \neq (3,3,2,2,1)$	$\lambda = (3,3,2,2,1)$	1, 1, 2, 2, 7
$\mathbf{D}_5$	$\lambda \neq (3,3,2,2)$	$\lambda = (3,3,2,2)$	1,1,2,2,5
$\mathbf{B}_6$	$\lambda \notin \{(5,3,2,2,1),(4,4,2,2,1),(3,3,2,2,1,1,1)\}$	$\lambda \in \{(5,3,2,2,1),(4,4,2,2,1),(3,3,2,2,1,1,1)\}$	{1, 1, 1, 2, 2, 7; 1, 1, 2, 2, 3, 6; 1, 1, 2, 2, 6, 7}
$\mathbf{D}_6$	$\lambda \notin \{(5,3,2,2),(3,3,2,2,1,1)\}$	$\lambda \in \{(5,3,2,2),(3,3,2,2,1,1)\}$	{1,1,1,2,2,5; 1,1,2,2,3,7}

TABLE 5. Conclusions for $\mathfrak{g}$ of type $\mathbf{B}_\ell$ or $\mathbf{D}_\ell$ with $\ell \leqslant 6$

7.3. **A counter-example.** From the rank 7, there are elements that do no satisfy the polynomial condition. The following example disconfirms a conjecture of Premet that any nilpotent element of a simple Lie algebra of classical type satisfies the polynomiality condition.

Example 7.8. Let e be a nilpotent element of $\mathfrak{so}(\mathbb{k}^{14})$ associated with the partition $(3,3,2,2,2,2)$. Then e does not satisfy the polynomiality condition.

In this case, $\ell = 7$ and let $q_1, \dots, q_7$ be as in Subsection 5.2. The degrees of ${}^eq_1, \dots, {}^eq_7$ are $1,2,2,3,4,5,3$ respectively. By a computation performed by Maple, one can show that ${}^eq_1, \dots, {}^eq_7$ verify the two following algebraic relations:

$$16\,{}^eq_3^2\,{}^eq_5{}^2 + {}^eq_4^4 - 8\,{}^eq_3\,{}^eq_5\,{}^eq_4^2 - 64\,{}^eq_3^3\,{}^eq_7{}^2 = 0, \qquad {}^eq_3\,{}^eq_6^2 - {}^eq_7^2\,{}^eq_4{}^2 = 0$$

Set:

$$r_i := \begin{cases} q_i & \text{if} \quad i = 1,2,3,4,7 \\ 16\,q_3^2q_5{}^2 + q_4^4 - 8\,q_3q_5q_4^2 - 64\,q_3^3q_7{}^2 & \text{if} \quad i = 5 \\ q_3q_6^2 - q_7^2q_4^2 & \text{if} \quad i = 6 \end{cases}$$

The polynomials $r_1, \dots, r_7$ are algebraically independent over $\mathbb{k}$ and

$$\mathrm{d}r_1 \wedge \cdots \wedge \mathrm{d}r_7 = 2q_3q_6\,(32q_3^2q_5 - 8q_3q_4^2)\,\mathrm{d}q_1 \wedge \cdots \wedge \mathrm{d}q_7$$

Moreover, ${}^er_5$ and ${}^er_6$ have degree at least 13 and ${}^e(2q_3q_6(32q_3^2q_5 - 8q_3q_4^2))$ has degree 15. Then, by Corollary 7.2, ${}^er_1, \dots, {}^er_7$ are algebraically independent since

$$\frac{1}{2}(\dim \mathfrak{g}^e + 7) + 15 = 37 = 1 + 2 + 2 + 3 + 3 + 26$$

and by Lemma 7.1,(ii) and (iii), ${}^er_5$ and ${}^er_6$ have degree 13.

A precise computation performed by Maple shows that ${}^er_3 = p_3^2$ for some p_3 in the center of $\mathfrak{g}^e$, ${}^er_4 = p_3p_4$ for some polynomial p_4 of degree 2 in $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$, ${}^er_5 = p_3^3\,{}^eq_7p_5$ for some polynomial p_5 of degree 7 in $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$, and ${}^er_6 = p_4\,{}^er_7p_6$ for some polynomial p_6 of degree 8 in $S(\mathfrak{g}^e)^{\mathfrak{g}^e}$. Setting $p_i := {}^er_i$ for $i = 1,2,7$, the polynomials $p_1, \dots, p_7$ are algebraically independent homogeneous polynomials of degree $1,2,1,2,7,8,3$ respectively. Let $\mathfrak{l}$ be a reductive factor of $\mathfrak{g}^e$. According to [C85, Ch.13],

$$\mathfrak{l} \simeq \mathfrak{so}_2(\mathbb{k}) \times \mathfrak{sp}_4(\mathbb{k}) \simeq \mathbb{k} \times \mathfrak{sp}_4(\mathbb{k}).$$

In particular, the center of $\mathfrak{l}$ has dimension 1. Let $\{x_1, \dots, x_{37}\}$ be a basis of $\mathfrak{g}^e$ such that x_{37} lies in the center of $\mathfrak{l}$ and such that $x_1, \dots, x_{36}$ are in $[\mathfrak{l}, \mathfrak{l}] + \mathfrak{g}^e_{\mathfrak{u}}$ with $\mathfrak{g}^e_{\mathfrak{u}}$ the nilpotent radical of $\mathfrak{g}^e$. Then p_2 is a polynomial in $\mathbb{k}[x_1, \dots, x_{37}]$ depending on x_{37}. As a result, by [DDV74, Thm. 3.3 and 4.5], the semiinvariant polynomials of $S(\mathfrak{g}^e)$ are invariant.

Claim 7.9. The algebra $\mathfrak{g}^e$ is nonsingular.

Proof. The space $\Bbbk^{14}$ is the ortogonal direct sum of two subspaces $\mathbb{V}_1$ and $\mathbb{V}_2$ of dimension 6 and 8 respectively and such that e, h, f are in $\overline{\mathfrak{g}} := \mathfrak{so}(\mathbb{V}_1) \oplus \mathfrak{so}(\mathbb{V}_2)$. Then $\overline{\mathfrak{g}}^e = \overline{\mathfrak{g}} \cap \mathfrak{g}^e$ is a subalgebra of dimension 21 containing the center of $\mathfrak{g}^e$. For p in $\mathrm{S}(\mathfrak{g}^e)$, denote by $\overline{p}$ its restriction to $\overline{\mathfrak{g}}^f$. The partition $(3,3,2,2,2,2)$ verifies the condition (1) of the proof of [Y06, §4, Lem. 3]. So, the proof of Lemma 5.14 remains valid, and the morphism

$$G_0^e \times \overline{\mathfrak{g}}^f \longrightarrow \mathfrak{g}^f, \qquad (g, x) \longmapsto g(x)$$

is dominant. As a result, for p in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$, the differential of $\overline{p}$ is the restriction to $\overline{\mathfrak{g}}^f$ of the differential of p. A computation performed by Maple proves that $\overline{p_3}^{10}$ is a great common divisor of $\mathrm{d}\overline{p_1} \wedge \cdots \wedge \mathrm{d}\overline{p_7}$ in $\mathrm{S}(\overline{\mathfrak{g}^e})$. If q is a greatest common divisor of $\mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_7$ in $\mathrm{S}(\mathfrak{g}^e)$, then q is in $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ since the semiinvaiant polynomials are invariant. So $q = p_3^d$ for some nonnegative integer d. One can suppose that $\{x_1, \ldots, x_{16}\}$ is a basis of the orthogonal complement of $\overline{\mathfrak{g}}^f$ in $\mathfrak{g}^e$. Then the Pfaffian of the matrix

$$\big([x_i, x_j],\ 1 \leqslant i, j \leqslant 16\big)$$

is in $\Bbbk^* p_3^8$ so that p_8^2 is a greatest common divisor of $\mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_7$ in $\mathrm{S}(\mathfrak{g}^e)$. Since

$$\deg p_1 + \cdots + \deg p_7 = 2 + 22 = 2 + \frac{1}{2}(\dim \mathfrak{g}^e + \ell),$$

we conclude that $\mathfrak{g}^e$ is nonsingular by Corollary 7.3,(ii). □

Claim 7.10. Suppose that $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is a polynomial algebra. Then for some homogeneous polynomials p_5' and p_6' of degrees at least 5 and at most 8 respectively, $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ is generated by p_1, p_2, p_3, p_4, p_5', p_6', p_7. Furthermore, the possible values for $(\deg p_5', \deg p_6')$ are $(5, 8)$ or $(6, 7)$.

Proof. Since the semiinvariants are invariants, by Claim 7.9 and Proposition 7.4,(ii), there are homogeneous generators $\varphi_1, \ldots, \varphi_\ell$ of $\mathrm{S}(\mathfrak{g}^e)^{\mathfrak{g}^e}$ such that

$$\deg \varphi_1 \leqslant \cdots \leqslant \deg \varphi_\ell,$$

and

$$\deg \varphi_1 + \cdots + \deg \varphi_\ell = \frac{1}{2}(\dim \mathfrak{g}^e + \ell) = 22.$$

According to [Mo06c, Thm. 1.1.8] or [Y06b], the center of $\mathfrak{g}^e$ has dimension 2. Hence, φ_1 and φ_2 has degree 1. Thereby, we can suppose that $\varphi_1 = p_1$ and $\varphi_2 = p_3$ since p_1 and p_3 are linearly independent elements of the center of $\mathfrak{g}^e$. Since p_2 and p_4 are homogneous elements of degree 2 such that $p_1, \ldots, p_4$ are algebraically indepent, φ_3 and φ_4 have degree 2 and we can suppose that $\varphi_3 = p_2$ and $\varphi_4 = p_4$. Since p_7 has degree 3, φ_5 has degree at most 3 and at least 2 since the center of $\mathfrak{g}^e$ has dimension 2. Suppose that φ_5 has degree 2. A contradiction is expected. Then

$$\deg \varphi_6 + \deg \varphi_7 = 22 - (1 + 1 + 2 + 2 + 2) = 14.$$

Moreover, since $p_1, \ldots, p_7$ are algebraically independent, φ_7 has degree at most 8 and φ_6 has degree at least 6. Hence p_7 is in the ideal of $\Bbbk[p_1, p_3, \varphi_3, \varphi_4, \varphi_5]$ generated by p_1 and p_3. But a computation shows that the restriction of p_7 to the nullvariety of p_1 and p_3 in $\mathfrak{g}^f$ is different from 0, whence the expected contradiction. As a result, φ_5 has degree 3 and

$$\deg \varphi_6 + \deg \varphi_7 = 13.$$

One can suppose $\varphi_5 = p_7$ and the possible values for $(\deg \varphi_6, \deg \varphi_7)$ are $(5, 8)$ and $(6, 7)$ since φ_7 has degree at most 8. □

Suppose that $S(\mathfrak{g})^{\mathfrak{g}}$ is a polynomial algebra. A contradiction is expected. Let p'_5 and p'_6 be as in Claim 7.10 and such that $\deg p'_5 < \deg p'_6$. Then $(\deg p'_5, \deg p'_6)$ equals $(5,8)$ or $(6,7)$. A computation shows that one can choose a basis $\{x_1,\dots,x_{37}\}$ of $\mathfrak{g}^e$ with $x_{37} = p_3$, with p_1, p_2, p_3, p_4, p_7 in $\Bbbk[x_3,\dots,x_{37}]$ and with p_5, p_6 of degree 1 in x_1. Moreover, the coefficient of x_1 in p_5 is a prime element of $\Bbbk[x_3,\dots,x_{37}]$, the coefficient of x_1 in p_6 is a prime element of $\Bbbk[x_2,\dots,x_{37}]$ having degree 1 in x_2, and the coefficient of x_1x_2 in p_6 equals $a^2p_3^2$ with a a prime homogeneous polynomial of degree 2 such that a, p_1, p_2, p_3, p_4 are algebraically independent. In particular, a is not invariant. If p'_5 has degree 5, then

$$p_5 = p'_5 r_0 + r_1$$

with r_0 in $\Bbbk[p_1,p_2,p_3,p_4]$ and r_1 in $\Bbbk[p_1,p_2,p_3,p_4,p_7]$ so that p'_5 has degree 1 in x_1, and the coefficient of x_1 in p_5 is the product of r_0 and the coefficient of x_1 in p'_5. But this is impossible. So, p'_5 has degree 6 and p'_6 has degree 7. We can suppose that $p'_6 = p_5$. Then

$$p_6 = p_5 r_0 + p'_6 r_1 + r_2$$

with r_0 homogeneous of degree 1 in $\Bbbk[p_1,p_3]$, r_1 homogeneous of degree 2 in $\Bbbk[p_1,p_2,p_3,p_4]$, and r_2 homogeneous of degree 8 in $\Bbbk[p_1,p_2,p_3,p_4,p_7]$. According to the above remarks on p_5 and the coefficient of x_1x_2 in p_6, r_1 is in $\Bbbk^* p_3^2$ since r_1 has degree 2.

For p in $S(\mathfrak{g}^e)$, denote by $\overline{p}$ its image in $S(\mathfrak{g}^e)/p_3S(\mathfrak{g}^e)$. A computation shows that for some u in $S(\mathfrak{g}^e)/p_3S(\mathfrak{g}^e)$,

$$\overline{p_5} = \overline{p_4}^2 u, \qquad \overline{p_6} = -\overline{p_4p_7}u.$$

Furthermore, $\overline{p_4}$ and $\overline{p_7}$ are different prime elements of $S(\mathfrak{g}^e)/p_3S(\mathfrak{g}^e)$ and the coefficient u_1 of x_1 in u is the product of two different polynomials of degree 1. The coefficient of x_1 in $\overline{p_6}$ is $u_1\overline{p_4}^2\overline{r_0}$ since

$$\overline{p_6} = \overline{p_5 r_0} + \overline{r_2}.$$

On the other hand, the coefficient of x_1 in $\overline{p_6}$ is $-u_1\overline{p_4p_7}$, whence the contradiction since r_0 has degree 1.

7.4. **Another result.** We are not able so far to deal with all even nilpotent elements of a Lie algebra of type **D** with odd rank. We can however state the following result. In what follows, we retain the notations of Subsection 5.3.

Theorem 7.11. *Let $\mathfrak{g} = \mathfrak{so}(\mathbb{V})$ and let e be a nilpotent element of $\mathfrak{g}$ associated with the sequence $\lambda = (\lambda_1,\dots,\lambda_k)$. Assume that λ verifies the condition $(*)$ and that $\lambda_1 = \dots = \lambda_{k'}$. Then there are algebraically independent elements $r_1,\dots,r_\ell$ in $S(\mathfrak{g})^{\mathfrak{g}}$ such that ${}^er_1,\dots,{}^er_\ell$ are algebraically independent.*

Proof. Let $s \in \{1,\dots,m\}$ and $i \in K_s$ written as $i = k_1+\dots+k_{s-1}+j$, with $j \in \{1,\dots,k_s\}$. For the sake of simplicity, set

$$k^*_{s-1} := k_1+\dots+k_{s-1}.$$

Assume that $j > k_s/2$ and let $R_j^{(s)}$ be as in Lemma 5.21. Since $R_j^{(s)}$ has degree j, for some polynomial $\hat{R}_j^{(s)}$,

$$(p_{k^*_{s-1}})^j R_j^{(s)}\Big(\frac{p_{k^*_{s-1}+1}}{p_{k^*_{s-1}}},\dots,\frac{p_{k^*_{s-1}+k_s/2}}{p_{k^*_{s-1}}}\Big) = \hat{R}_j^{(s)}(p_{k^*_{s-1}}, p_{k^*_{s-1}+1},\dots,p_{k^*_{s-1}+k_s/2}).$$

Then by Lemma 5.14 and Lemma 5.21,

$$(p_{k^*_{s-1}})^{j-1}\, p_{k^*_{s-1}+j} = \hat{R}_j^{(s)}(p_{k^*_{s-1}}, p_{k^*_{s-1}+1},\dots,p_{k^*_{s-1}+k_s/2}).$$

Define polynomials $r_1,\dots,r_\ell$ of $S(\mathfrak{g})^{\mathfrak{g}}$ as follows.

- If $k' < k$, then
  * for $l \in \{1,\dots,\ell\} \smallsetminus \{\nu_i,\ i \in (K_1\cup\ldots\cup K_m)\smallsetminus(I_1\cup\ldots\cup I_m)\}$, set $r_l := q_l$,

* for $i \in (K_1 \cup \cdots \cup K_m) \smallsetminus (I_1 \cup \cdots \cup I_m)$, set

$$r_{\nu_i} := (q_{\nu_{k^*_{s-1}}})^{j-1} q_{\nu_{k^*_{s-1}+j}} - \hat{R}^{(s)}_j (q_{\nu_{k^*_{s-1}}}, q_{\nu_{k^*_{s-1}+1}}, \dots, q_{\nu_{k^*_{s-1}+k_s/2}}).$$

- If $k' = k$, then
  * for $l \in \{1, \dots, \ell\} \smallsetminus \{\nu_i,\ i \in (K_1 \cup \ldots \cup K_m) \smallsetminus (I_1 \cup \ldots \cup I_m)\}$, set $r_l := q_l$,
  * for $i \in (K_1 \cup \cdots \cup K_m) \smallsetminus (I_1 \cup \cdots \cup I_m)$, set

$$r_{\nu_i} := (q_{\nu_{k^*_{s-1}}})^{j-1} q_{\nu_{k^*_{s-1}+j}} - \hat{R}^{(s)}_j (q_{\nu_{k^*_{s-1}}}, q_{\nu_{k^*_{s-1}+1}}, \dots, q_{\nu_{k^*_{s-1}+k_s/2}})$$

  if $\nu_i \neq \ell$, that is $i \neq k$, and set

$$r_{\nu_k} := (q_{\nu_{k^*_{m-1}}})^{k_m-1} (q_{\nu_k})^2 - \hat{R}^{(m)}_{k_m} (q_{\nu_{k^*_{m-1}}}, q_{\nu_{k^*_{m-1}+1}}, \dots, q_{\nu_{k^*_{m-1}+k_m/2}})$$

  otherwise.

Then

$$\mathrm{d}r_1 \wedge \cdots \wedge \mathrm{d}r_\ell = p\,(\mathrm{d}q_1 \wedge \cdots \wedge \mathrm{d}q_\ell) \quad \text{where} \quad p = \prod_{s=1}^{m} (q_{\nu_{k^*_{s-1}}})^{k_{s/2}+\cdots+k_s-1}.$$

Hence,

$$\deg {}^e p = \sum_{s=1}^{m} (k_1 + \cdots + k_{s-1})(k_{s/2} + \cdots + k_s - 1).$$

Let δ^* be the sum of the degrees of the polynomials ${}^e r_1, \dots, {}^e r_\ell$. By construction, one has

$$\begin{aligned} \delta^* &\geqslant \sum_{i=1}^{\ell} \deg {}^e q_i + \sum_{s=1}^{m} (k_1 + \cdots + k_{s-1})(k_{s/2} + \cdots + k_s - 1) + \mathrm{card}((K_1 \cup \cdots \cup K_m) \smallsetminus (I_1 \cup \cdots \cup I_m)) \\ &= \sum_{i=1}^{\ell} \deg {}^e q_i + \deg {}^e p + \frac{k'}{2}. \end{aligned}$$

On the other hand, by Remark 5.24, one has $\dim \mathfrak{g}^e + \ell - 2 \sum_{i=1}^{\ell} \deg {}^e q_i = k'$. As a result,

$$\sum_{i=1}^{\ell} \deg {}^e r_i \geqslant \deg {}^e p + \frac{1}{2}(\dim \mathfrak{g}^e + \ell),$$

whence the theorem by Lemma 7.1,(ii) and (iii). □

7.5. **A conjecture.** All examples of good elements we achieved satisfy the hypothesis of Theorem 4.1. This leads us to formulate a conjecture.

Conjecture 7.12. *Let $\mathfrak{g}$ be a simple Lie algebra and let e be a good nilpotent of $\mathfrak{g}$. Then for some homogeneous generators $q_1, \dots, q_\ell$ of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, the polynomial functions ${}^e q_1, \dots, {}^e q_\ell$ are algebraically independent. In other words, the converse implication of Theorem 4.1 holds.*

References

[BD] Alexander Beilinson and Vladimir Drinfeld, *Quantization of Hitchins integrable system and Hecke eigensheaves*, preprint available on http://www.math.utexas.edu/users/benzvi/BD/hitchin.pdf.

[B91] A.V. Bolsinov, *Commutative families of functions related to consistent Poisson brackets*, Acta Applicandae Mathematicæ, **24** (1991), **n°1**, 253–274.

[BG97] K.A. Brown and K.R. Goodearl, *Homological Aspects of Noetherian PI Hopf Algebras and Irreducible Modules of Maximal Dimension*, Journal of Algebra , **198** (1997), 240–265.

[BK06] J. Brundan and A. Kleshchev, *Shifted Yangians and finite W-algebras*, Adv. Math. **200** (2006), **n°1**, 136–195.

[C85] R. W. Carter, *Finite groups of Lie type: Conjugacy classes and complex characters*, Wiley, New York, 1985.

[CM10] J.-Y. Charbonnel and A. Moreau, *The index of centralizers of elements of reductive Lie algebras*, Documenta Mathematica, **15** (2010), 387-421.

[CMc93] D. Collingwood and W.M. McGovern, *Nilpotent orbits in semisimple Lie algebras*, Van Nostrand Reinhold Co. New York, **65** (1993).

[DeG08] W.A. DeGraaf, *Computing with nilpotent orbits in simple Lie algebras of exceptional type*, London Math. Soc. (2008), 1461–1570.

[Di74] J. Dixmier, *Algèbres enveloppantes*, Gauthier-Villars (1974).

[DDV74] J. Dixmier, M. Duflo and M. Vergne, *Sur la représentation coadjointe d'une algèbre de Lie*, Composition Mathematica, **29**, (1974), 309–323.

[DV69] M. Duflo and M. Vergne, *Une propriété de la représentation coadjointe d'une algèbre de Lie*, C.R.A.S. Paris (1969).

[GG02] W.L. Gan and V. Ginzburg, *Quantization of Slodowy slices*, Int. Math. Res. Not., (2002), 243–255.

[JS10] A. Joseph and D. Sharir, *Polynomiality of invariants, unimodularity and adapted pairs*, Transformation Groups, **15**, (2010), 851–882.

[McR01] J.C. McConnell and J.C. Robson, *Noncommutative Noetherian Rings*, Graduate Studies in Mathematics (2001), **n° 30**, American Mathematical Society Providence, Rhode Island.

[Ma86] H. Matsumura, *Commutative ring theory* Cambridge studies in advanced mathematics (1986), **n°8**, Cambridge University Press, Cambridge, London, New York, New Rochelle, Melbourne, Sydney.

[Me88] M.L. Mehta, *Basic sets of invariant polynomials for finite reflection groups*, Comm. Algebra **16** (1988), **n°5**, 1083–1098.

[Mo06c] A. Moreau, *Quelques propriétés de l'indice dans une algèbre de Lie semi-simple*, PhD Thesis (2006), available on http://www.institut.math.jussieu.fr/theses/2006/moreau/.

[Mu88] D. Mumford, *The Red Book of Varieties and Schemes*, Lecture Notes in Mathematics (1988), **n° 1358**, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo.

[Pa03] D.I. Panyushev, *The index of a Lie algebra, the centralizer of a nilpotent element, and the normaliser of the centralizer*, Math. Proc. Camb. Phil. Soc., **134** (2003), 41–59.

[PPY07] D.I. Panyushev, A. Premet and O. Yakimova, *On symmetric invariants of centralizers in reductive Lie algebras*, Journal of Algebra **313** (2007), 343–391.

[Pr02] A. Premet, *Special transverse slices and their enveloping algebras*, Advances in Mathematics **170** (2002), 1–55.

[R63] M. Rosenlicht, *A remark on quotient spaces*, Anais da Academia brasileira de ciencias **35** (1963), 487–489.

[SS70] T.A. Springer and R. Steinberg, *Conjugacy classes.* In : A. Borel et al., *Seminar on algebraic groups and related finite groups*, Lecture Notes in Mathematics (1970), **n°131**, Springer-Verlag, Berlin, Heidelberg, New York.

[T12] L. Topley, *Symmetric Invariants of centralizers in Classical Lie Algebras and the KW1 Conjecture*, preprint arxiv.org/abs/1108.2306.

[V72] F.D. Veldkamp, *The center of the universal enveloping algebra of a Lie algebra in characteristic p*, Annales Scientifiques de L'École Normale Supérieure **5** (1972), 217–240.

[Y06] O. Yakimova, *The index of centralisers of elements in classical Lie algebras*, Functional Analysis and its Applications **40** (2006), 42–51.

[Y06b] O. Yakimova, *Centers of centralisers in the classical Lie algebras*, preprint available on http://www.mccme.ru/ yakimova/center/center.pdf (2006).

[Y07] O. Yakimova, *A counterexample to Premet's and Joseph's conjecture*, Bulletin of the London Mathematical Society **39** (2007), 749–754.

[Y09] O. Yakimova, *Surprising properties of centralisers in classical Lie algebras*, Ann. Inst. Fourier (Grenoble) **59** (2009), **n°3**, 903–935.

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